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Asymptotic Telegrapher's Equation (P_1) Approximation for the Transport Equation

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Abstract—*The diffusion approximation for the Boltzmann (transport) equation suffers from several disadvantages. First, the diffusion approximation succeeds in describing the particle density only if it is isotropic, or close to isotropic. This feature causes the diffusion approximation to be quite accurate for highly isotropically scattering media but to yield poor agreement with the exact solution for the particle density in the case of nonisotropic behavior. To handle general media, the asymptotic diffusion approximation was first developed in the 1950s. The second disadvantage is that the parabolic nature of the diffusion equation predicts that particles will have an infinite velocity; particles at the tail of the distribution function will show up at infinite distance from a source in time $t = 0+$. The classical P_1 approximation (which gives rise to the Telegrapher's equation) has a finite particle velocity but with the wrong value, namely, $v/\sqrt{3}$. In this paper we develop a new approximation from the asymptotic solution of the time-dependent Boltzmann equation, which includes the correct eigenvalue of the asymptotic diffusion approximation and the (almost) correct time behavior (such as the particle velocity), for a general medium.*

I. INTRODUCTION

The Boltzmann equation, or the linear transport equation, describes the local density of particles traveling inside a medium with interactions between the particles and the medium. The equation is an integrodifferential equation, and the local particle density depends on position, time, and velocity (or, alternatively, energy and direction of motion).[1–5]

The energy dependence is usually modeled with the multigroup approximation (also called the multienergy approximation), where the energy-space divides into discrete energy groups. Thus, this approximation reduces the problem from an energy-dependent transport equation to G monoenergetic equations (where G is the number of energy groups), which couple through the source term (all the transitions between energy groups are due to the source term). Therefore, the effort is focused on solving the monoenergetic Boltzmann equation and especially on finding the angular dependency.

There are several well-known approaches to handle the angular dependence of the Boltzmann equation. The most popular are the P_N approximation (where the angular flux is decomposed into a Legendre series), the S_N method (also called the discrete ordinates method where the scalar flux is discretized into several discrete directions), and a family of diffusion approximations[1–4] (the first two approximations reach the exact solution when $N \to \infty$). The classic diffusion approximation (also called the Eddington approximation[4]) is developed by taking the angular dependence to be isotropic, or almost isotropic (alternatively, it can be developed from a simple derivation from the P_N approximation with $N = 1$). Naturally, the diffusion approximation describes the particle density

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well only when it is isotropic, for example, in a highly (isotropic) scattering medium. When the particle density (or flux) is far from isotropic, as in a streaming problem, there is poor agreement with the classical diffusion approximation.

For this reason, the asymptotic diffusion approximation was first developed in the 1950s by Frankel and Nelson and expanded by Case et al. and by Davison (also called the rigorous approximation or the extrapolated end-point method).[4–8] In this approximation, a modified Fick's law is derived from an asymptotic treatment of the exact time-independent Boltzmann equation (in general geometry[4,8]). The modified diffusion coefficient is material dependent and recovers the exact eigenvalues of the asymptotic solution for general media. Thus, the asymptotic diffusion approximation succeeds in predicting the scalar flux well when the asymptotic part is dominant. The asymptotic diffusion approximation enables the solution of time-independent problems, such as the critical mass of bare reactor problems in neutronics.[2]

In time-dependent problems, the diffusion equation (both classical or asymptotic) fails to describe the front of streaming generating from a source, even for a highly scattering medium (due to the parabolic nature of the diffusion equation). The diffusion equation produces particles at an infinite distance from the source at time $t = 0+$; i.e., some particles are allowed to have infinite particle velocity. A full (time-dependent) P_1 approximation for the Boltzmann equation, which gives rise to the Telegrapher's equation,[3] replaces this feature of an infinite velocity with a wrong finite velocity $v/\sqrt{3}$. We note that the Telegrapher's equation is widely used in many fields, in addition to transport.[9]

Much effort has been expended in solving this problem in the diffusion-like regime because of the relatively simple, well-known treatment and the great success in solving diffusion equations in multidimensional problems (instead of the full Boltzmann equation). A large variety of approximate methods was offered, such as flux-limiter solutions[10–15] or Eddington factor approximations[13,15] (in P_1 form), both being ad hoc and semiphysical solutions. These approximations usually use flux-dependent (or more correctly, gradient-flux-dependent) diffusion coefficients (or Eddington factors) causing the diffusion equation (or P_1 equations in the Eddington factor approximation case) to be nonlinear. Despite the great success and the wide use of these approximations, the nonlinearity of the equation makes finding the solution much more difficult, and analytic solutions are seldom available. It should be noted that in one-dimensional problems, a certain P_1 approximation can be derived that predicts the exact transport behavior, based on the forward-backward approximation[16,17] (analogous to the S_2 approximation with $\mu = \pm 1$). Unfortunately, this derivation has no analogy in multidimensional problems.

This work attempts to follow the rationale of the asymptotic diffusion approximation that was derived from the asymptotic behavior of the time-independent Boltzmann equation and to apply it to time-dependent problems. The motivation is to develop modified (linear) P_1 equations (instead of a modified Fick's law in the time-independent case), derived from the asymptotic behavior of the time-dependent Boltzmann equation, that will reproduce both the asymptotic time-independent eigenvalue and the time behavior, such as correct particle velocity. This procedure is based on the time-dependent treatment presented in Case and Zweifel,[7] Case et al.,[8] and Kuscer and Zweifel.[18] The major advantage of this approach is the ability to predict the asymptotic behavior of the time-dependent flux by solving only the Telegrapher's equation (or P_1 equations) instead of solving the full Boltzmann equation. Although it is not as simple as a diffusion equation, it is still much simpler than solving the full Boltzmann equation, especially in multidimensional problems. This procedure allows us to exploit the major progress that has been achieved in solving the Telegrapher's equation,[19–24] both analytical and numerical (for multidimensional problems). It should be noted that an important similar attempt was introduced recently,[25] based on a development that differs from Case's "classical" approach, and we refer to it below. For charged particles in the isotropic case only, we refer also to Refs. 26 and 27.

The work is structured as follows. First, we present the main definitions and derive the basic equations that are used in this work (Sec. II). Section II is crucial to explain the rationale of the new asymptotic Telegrapher's equation approximation. Next, Sec. III introduces the derivation of the asymptotic Telegrapher's equation (P_1) approximation from the asymptotic analysis of the exact Boltzmann equation (for general multidimensional geometry). Section IV shows the achievements of this new approximation in describing the time behavior of the particle density. A brief concluding discussion is given in Sec. V.

II. DEFINITIONS AND BASIC DERIVATIONS

The monoenergetic linear transport equation in the "flux form" can be written as (for example, see Ref. 1)

$$\frac{1}{v}\frac{\partial\psi(\vec{r},\hat{\Omega},t)}{\partial t} + \hat{\Omega}\cdot\vec{\nabla}\psi(\vec{r},\hat{\Omega},t) + \Sigma_t(\vec{r})\psi(\vec{r},\hat{\Omega},t) = \int_{4\pi} d\hat{\Omega}'\Sigma_s(\vec{r},\hat{\Omega}\cdot\hat{\Omega}')\psi(\vec{r},\hat{\Omega}',t) + Q(\vec{r},\hat{\Omega},t) \ , \tag{1}$$

where

$\psi(\vec{r},\hat{\Omega},t) = vn(\vec{r},\hat{\Omega},t)$ = angular flux [v is the particle velocity and $n(\vec{r},\hat{\Omega},t)$ is the local particle density], which depends on the position $\vec{r}$, the time t, and the particle direction of motion $\hat{\Omega}$

$\Sigma_t(\vec{r}) = \Sigma_a(\vec{r}) + \Sigma_s(\vec{r})$ = total cross section, which depends on the position, where $\Sigma_a(\vec{r})$ is the absorbing cross section and $\Sigma_s(\vec{r})$ is the scattering cross section $[\Sigma_s(\vec{r}) = \int_{4\pi} d\hat{\Omega}'\, \Sigma_s(\vec{r},\hat{\Omega}\cdot\hat{\Omega}')]$

$Q(\vec{r},\hat{\Omega},t)$ = source term that can describe either outer or inner sources (as in fission).

There are several ways to derive the diffusion equation from the Boltzmann equation. The most straightforward is presented here, and this presentation is crucial to understanding the rationale of the new approximation later. First, integrating $\int_{4\pi} d\hat{\Omega}$ over the Boltzmann equation yields

$$\frac{1}{v}\frac{\partial\phi(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot\vec{J}(\vec{r},t) + \Sigma_a(\vec{r})\phi(\vec{r},t) = Q^{(0)}(\vec{r},t) \ , \tag{2}$$

where

$\phi(\vec{r},t) \equiv \int_{4\pi} \psi(\vec{r},\hat{\Omega},t)\, d\hat{\Omega}$ = zero moment of the angular flux $\psi(\vec{r},\hat{\Omega},t)$ called the scalar flux

$\vec{J}(\vec{r},t) \equiv \int_{4\pi} \psi(\vec{r},\hat{\Omega},t)\hat{\Omega}\, d\hat{\Omega}$ = first moment of the angular flux called the current density

$Q^{(0)}(\vec{r},t) \equiv \int_{4\pi} Q(\vec{r},\hat{\Omega},t)\, d\hat{\Omega}$.

Equation (2) is the zero moment of the Boltzmann equation. It is an exact equation and well known as the conservation law. Next, operating $\int_{4\pi} \hat{\Omega} d\hat{\Omega}$ over the Boltzmann equation yields

$$\frac{1}{v}\frac{\partial\vec{J}(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot\int_{4\pi}\hat{\Omega}\hat{\Omega}\psi(\vec{r},\hat{\Omega},t)\, d\hat{\Omega} + \Sigma_t(\vec{r})\vec{J}(\vec{r},t) = \bar{\mu}_0\Sigma_s(\vec{r})\vec{J}(\vec{r},t) + Q^{(1)}(\vec{r},t) \ , \tag{3}$$

where $\bar{\mu}_0$ is the average scattering angle cosine defined by

$$\bar{\mu}_0 = \langle\hat{\Omega}\cdot\hat{\Omega}'\rangle = \frac{1}{4\pi\Sigma_s(\vec{r})}\int_{4\pi} d\hat{\Omega}\int_{4\pi} d\hat{\Omega}'\,\hat{\Omega}\cdot\hat{\Omega}'\Sigma_s(\vec{r},\hat{\Omega}\cdot\hat{\Omega}') \tag{4}$$

and $Q^{(1)}(\vec{r},t) \equiv \int_{4\pi} Q(\vec{r},\hat{\Omega},t)\hat{\Omega}\, d\hat{\Omega}$. We can see that Eq. (3) contains a term of the second moment of the angular flux. So far, the derivation has been exact. It is important to notice that each moment of the Boltzmann equation contains a higher moment of the angular flux. Thus, in order to have a finite number of equations, one must have an intelligent approximation for the additional moment.

As said before, the diffusion approximation describes the particle flux in the isotropic or almost isotropic case. The first approximation that needs to be done is assuming that the angular flux can be written as a sum of its two first moments:

$$\psi(\vec{r},\hat{\Omega},t) \cong \frac{1}{4\pi}\phi(\vec{r},t) + \frac{3}{4\pi}\vec{J}(\vec{r},t)\cdot\hat{\Omega} \ , \tag{5}$$

assuming that $\phi(\vec{r},t) \gg |\vec{J}(\vec{r},t)|$. If the angular flux can be written in the form of Eq. (5), we can approximate the term containing the second moment of the angular flux in Eq. (3) as

$$\vec{\nabla}\cdot\int_{4\pi}\hat{\Omega}\hat{\Omega}\psi(\vec{r},\hat{\Omega},t)\, d\hat{\Omega} \cong \frac{\vec{\nabla}\phi(\vec{r},t)}{3} \ . \tag{6}$$

Second, we assume that the particle source is isotropic, so that $Q^{(1)}(\vec{r},t) = 0$. This approximation is necessary for the derivation of both the diffusion and the Telegrapher's equation approximations. Substituting Eq. (6) in Eq. (3) gives the next approximate equation:

$$\frac{3}{v}\frac{\partial\vec{J}(\vec{r},t)}{\partial t} + \vec{\nabla}\phi(\vec{r},t) + 3\Sigma_{tr}(\vec{r})\vec{J}(\vec{r},t) = 0 \ , \tag{7}$$

where $\Sigma_{tr}(\vec{r}) \equiv \Sigma_t(\vec{r}) - \bar{\mu}_0\Sigma_s(\vec{r})$ is called the transport cross section and gives a natural definition of the extended approximation.[28] When the scattering cross section is isotropic, we get $\Sigma_{tr}(\vec{r}) = \Sigma_t(\vec{r})$. Equations (2) and (7) constitute a closed set of equations for the scalar flux $\phi(\vec{r},t)$ and the current density $\vec{J}(\vec{r},t)$. Equations (2) and (7) are also called the P_1 approximation.

For a full treatment of the angular distribution of the particle density, one can use the P_N approximation. In this approximation, the angular flux is decomposed into a Legendre series for one-dimensional problems or into the spherical harmonics series for the general geometry (multidimensional) case. In a general geometry, the form of the P_N equation may be complicated (for $N > 1$), even for two- or three-dimensional Cartesian geometry.[1] The only exception is the $N = 1$ case (the P_1 case), when the two equations that define the P_1 approximation for any general geometry are identical to Eqs. (2) and (7) (Ref. 1).

If the derivative of the particle current with respect to time is negligible [i.e., $(1/|\vec{J}(\vec{r},t)|)(\partial|\vec{J}(\vec{r},t)|/\partial t) \ll v\Sigma_t$], Eq. (7) takes a Fick's law form (which gives rise to the diffusion approximation):

$$\vec{J}(\vec{r},t) = -D(\vec{r})\vec{\nabla}\phi(\vec{r},t) \ , \tag{8}$$

with a diffusion coefficient $D(\vec{r}) \equiv 1/3\Sigma_t(\vec{r})$. We used Σ_t (an isotropic scattering case) for simplicity, but now and during the entire paper, we can replace Σ_t with Σ_{tr}, for a linearly anisotropic scattering case. Substituting Eq. (8) in Eq. (2) (the conservation law) yields the diffusion equation:

$$\frac{1}{v}\frac{\partial\phi(\vec{r},t)}{\partial t} - \vec{\nabla}\cdot(D(\vec{r})\vec{\nabla}\phi(\vec{r},t)) + \Sigma_a\phi(\vec{r},t) = Q^{(0)}(\vec{r},t) \ . \tag{9}$$

The diffusion equation can be solved via many techniques in arbitrary dimension.

We now return to the full P_1 equations [Eqs. (2) and (7), *including* the derivative of J with respect to the time]. Simple derivation of Eqs. (2) and (7), assuming a uniform medium and a time-independent source term, gives rise to the hyperbolic Telegrapher's equation:

$$\left[\frac{1}{v}\right]\frac{\partial^2\phi(\vec{r},t)}{\partial t^2} - \left[\frac{v}{3}\right]\nabla^2\phi(\vec{r},t) + [\Sigma_t + \Sigma_a]\frac{\partial\phi(\vec{r},t)}{\partial t} + [v\Sigma_t\Sigma_a]\phi(\vec{r},t) = v\Sigma_t Q(\vec{r}) \ . \tag{10}$$

The assumption of a time-independent source is not mandatory, but it simplifies our equation by avoiding terms of order $\partial Q^{(0)}(\vec{r},t)/\partial t$ (Refs. 29 and 30). (Henceforth, in Sec. IV.B, we include time-dependent sources, for comparison with exact semianalytical published solutions.) To derive a Telegrapher's equation form for a nonuniform medium, we have to assume that $(1/v)\vec{\nabla}(1/\Sigma_t(\vec{r}))\cdot(\partial\vec{J}(\vec{r},t)/\partial t)$ is small, which is a reasonable approximation.[4] In any case, the main conclusions of this work remain the same even if we do not neglect this term, but then the equations will stay in the form of P_1 equations and not in the form of a single Telegrapher's equation.

To demonstrate the difference between the Telegrapher's equation (P_1) and the diffusion approximation, we introduce a one-dimensional infinite-slab problem with a time-independent delta source in the middle (at $x = 0$), i.e., $Q(x) = Q_0\delta(x)$ (this solution of the scalar flux may be used as the Green function for a general source on space). The initial conditions are chosen simply as $\phi(x,0) = 0$ and $(\partial\phi(x,t)/\partial t)|_{t=0} = 0$.

First, we examine the diffusion solution for this problem. The solution of the scalar flux, using standard techniques to solve differential equations (applying the Fourier transform on space and the Laplace transform on time) is[31 a]

$$\phi(x,t) = \frac{Q_0}{4}\sqrt{\frac{3\Sigma_t}{\Sigma_a}} \times \left[e^{-\sqrt{3\Sigma_t\Sigma_a}|x|}\,\mathrm{erfc}\left(\sqrt{\frac{3\Sigma_t}{4vt}}|x| - \sqrt{\Sigma_a vt}\right) - e^{\sqrt{3\Sigma_t\Sigma_a}|x|}\,\mathrm{erfc}\left(\sqrt{\frac{3\Sigma_t}{4vt}}|x| + \sqrt{\Sigma_a vt}\right)\right] , \tag{11}$$

where $\mathrm{erfc}(x') \equiv (2/\sqrt{\pi})\int_{x'}^{\infty} e^{-u^2}\,du = 1 - \mathrm{erf}(x')$. According to Eq. (11), some particles are allowed to have infinite particle velocities, existing on the tail of the scalar flux to infinity in time $t = 0+$. At very long times the solution presented at Eq. (11) tends to the well-known steady-state solution of the diffusion approximation for the Boltzmann equation[1,2]:

$$\phi(x, t\to\infty) = \frac{Q_0}{2}\sqrt{\frac{3\Sigma_t}{\Sigma_a}}e^{-\sqrt{3\Sigma_t\Sigma_a}|x|} \ . \tag{12}$$

Second, the scalar flux resulting from the Telegrapher's equation (applying the Fourier transform on space and the Laplace transform on time) is[31 b]

$$\phi(x,t) = \frac{\sqrt{3}v\Sigma_t Q_0}{2}\int_0^t e^{-(v/2)(\Sigma_a+\Sigma_t)u} \cdot I_0\left[\frac{v(\Sigma_a - \Sigma_t)}{2}\sqrt{u^2 - 3\frac{|x|^2}{v^2}}\right] \cdot H\left(u - \sqrt{3}\frac{|x|}{v}\right) du \ , \tag{13}$$

where

$I_n(x')$ = modified Bessel function of the first kind of order n

$H(x,t)$ = Heaviside step function

u = dummy variable.

In the purely absorbing medium case, Eq. (13) has this form:

$$\phi(x,t) = \frac{\sqrt{3}Q_0}{2}(e^{-\sqrt{3}\Sigma_a|x|} - e^{-v\Sigma_a t})H\left(t - \sqrt{3}\frac{|x|}{v}\right) \ . \tag{14}$$

We can clearly see from Eq. (13) or Eq. (14) that using the Telegrapher's equation yields a finite velocity, but a wrong one, $v/\sqrt{3}$. We note that the solution of the scalar flux in Eq. (13) is very similar to the solutions for the

[a]For the diffusion case we use the transform-pair on p. 246, No. 16, of Ref. 31.

[b]For the Telegrapher's equation we use the pair on p. 253, No. 63, of Ref. 31.

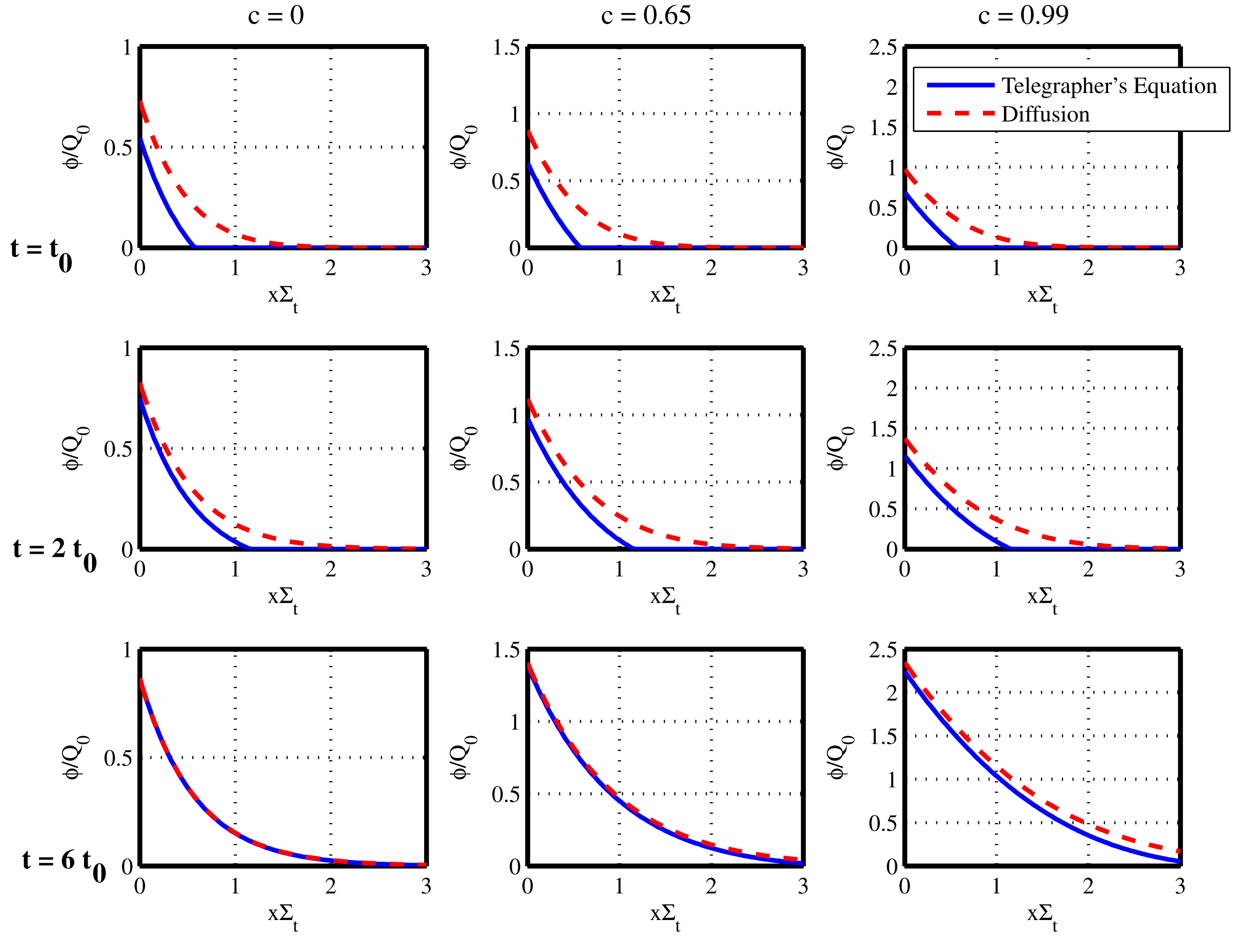

Fig. 1. The scalar flux in a source problem in a purely absorbing medium ($c = 0$) resulting from the Telegrapher's equation (solid lines) using Eq. (13) [or alternatively, Eq. (14)] and in the diffusion approximation [Eq. (11): dashed lines] at different times, with $c = 0.65$ using Eq. (13), and in a highly scattering medium ($c \approx 1$); $t_0 \equiv (v\Sigma_t)^{-1}$, and $t = 6t_0$ represents very long times.

Telegrapher's equation introduced in a large variety of works, with the same features.[9,19–24,30,32,33] At long times ($t \to \infty$), solving the integral in Eq. (13) (Ref. 34[c]) reproduces the diffusion solution at long times [Eq. (12)] exactly.

In Fig. 1 we show the scalar flux for three cases, using the definition of the albedo—the mean number of secondary particles emitted per collision event—for convenience:

$$c = \frac{\Sigma_s}{\Sigma_t} = \frac{\Sigma_s}{\Sigma_a + \Sigma_s} \; . \tag{15}$$

For purely absorbing media, $c = 0$, and for a highly scattering medium, $c \approx 1$. The third case includes both absorbing and scattering: $c = 0.65$ [in some problems c can be >1, like in multiplying media or source problems; then, the definition in Eq. (15) for the albedo should be extended].

[c] Integral on p. 722, session 6.646, No. 2, of Ref. 34.

In Fig. 1 we see the main features of the solution for the Telegrapher's equation: Eq. (13) [and for a purely absorbing medium, also Eq. (14)]. In early times, the scalar flux is naturally flux limited unlike the diffusion solution, but with the wrong particle velocity, $v/\sqrt{3} \approx 0.58v$ (for all values of c). When $c \neq 1$, the steady-state eigenvalue is also wrong (both in the diffusion and the Telegrapher's equation approximations), with the same magnitude of error [for example, in $c = 0$ at $t \to \infty$, there is a $\sqrt{3}$ factor in the exponent space decay in Eq. (14) compared to the asymptotic solution]. At long times, the Telegrapher's solution tends to the diffusion solution, as explained.

III. DERIVING THE ASYMPTOTIC TELEGRAPHER'S EQUATION APPROXIMATION

After presenting the capabilities and the disadvantages of the classic Telegrapher's equation (P_1), we attempt to develop an approximation that will include the correct time behavior, specifically, the correct particle velocity for general media. To derive the approximation, we first have to describe briefly the derivation of the asymptotic steady-state diffusion theory. Next, we show the basic rationale of the new asymptotic Telegrapher's approximation (time dependent), and then, we introduce a rigorous derivation of a modified Telegrapher's equation (or modified P_1 equations) from the exact time-dependent Boltzmann equation.

III.A. The Asymptotic (Steady-State) Diffusion Approximation

The steady-state classical diffusion approximation can be developed from the two equations comprising the classical steady-state P_1 equations, as introduced earlier. The first one is an exact one, the conservation law [Eq. (2)], and the second one is an approximate one, called the Fick's law [Eq. (7)]. The basic rationale for developing the asymptotic steady-state diffusion equation is to obtain a modified Fick's law derived from asymptotic solution of the exact time-independent Boltzmann equation, in addition to the exact conservation law. By "asymptotic" we mean far from sources or boundaries between two different media. Thus, the solution of such asymptotic diffusion approximation should provide the asymptotic steady-state behavior of the transport equation.

For simplicity, the derivation here is presented for the monoenergetic isotropic homogeneous one-dimensional slab case. This analysis appears in many works, such as Refs. 4, 7, and 8. The monoenergetic, one-dimensional slab geometry, time-independent Boltzmann equation in terms of the total cross section and the albedo c is

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \Sigma_t \psi(x,\mu) = \frac{c\Sigma_t}{2}\int_{-1}^{1} \psi(x,\mu)\, d\mu \ . \quad (16)$$

Equation (16) may be solved by separation of variables of the angular flux $\psi(x,\mu)$: space (x) and μ (the cosine of the angle between the particle and the x direction) of this type:

$$\psi(x,\mu) = e^{-\varkappa\Sigma_t x} f(\mu) \ . \quad (17)$$

Substituting Eq. (17) in Eq. (16) yields a closed equation for $\varkappa$, after some simple analysis:

$$\frac{2}{c} = \frac{1}{\varkappa_0} \ln\left(\frac{1+\varkappa_0}{1-\varkappa_0}\right) \ . \quad (18)$$

Equation (18) yields only two solutions for $\varkappa$ in the range of $|\varkappa| < 1$: $\pm\varkappa_0$, and they are called the asymptotic solutions. The values of $\varkappa_0$ as a function of c are tabulated and approximated extensively in Ref. 8. The asymptotic solution for the angular flux $\psi(x,\mu)$ is a superposition of the asymptotic solutions:

$$\psi(x,\mu) = \frac{c}{2} A_0 \frac{e^{\varkappa_0\Sigma_t x}}{1+\mu\varkappa_0} + \frac{c}{2} B_0 \frac{e^{-\varkappa_0\Sigma_t x}}{1-\mu\varkappa_0} \ . \quad (19)$$

The solution of $\varkappa_0(c)$ gives us the correct steady-state asymptotic eigenvalue. In the purely absorbing case, $\varkappa_0 = 1$, so the erroneous $\sqrt{3}$ factor vanishes. As $c \to 1$, $\varkappa_0 \to 0$, as expected.

There is a range of solutions that satisfy $|\varkappa| > 1$, each for different μ, which is defined in the range of $|\mu| < 1$. These solutions are called transient solutions because they decay relatively fast, compared to the two-solution $\varkappa = \pm\varkappa_0$, which decay slowly, and are valid near sources or boundaries between media. The exact transport solution is a combination of the asymptotic solutions and the transient solutions.

Calculating the first two moments of the asymptotic angular flux $\psi(x,\mu)$, $\phi(x)$ (the scalar flux) and $J(x)$, yields a relation between these two moments in the form of a modified Fick's law:

$$J(x) = -D(c,\Sigma_t)\cdot \frac{\partial \phi(x)}{\partial x} \ , \quad (20)$$

with a modified diffusion coefficient that depends on the properties of the medium (the albedo c), instead of the constant classical diffusion coefficient $D = 1/3\Sigma_t$:

$$D(c,\Sigma_t) \equiv \frac{1-c}{\varkappa_0^2(c)\Sigma_t} \equiv D_0(c)\frac{1}{\Sigma_t} \ , \quad (21)$$

where $D_0(c) \equiv (1-c)/\varkappa_0^2(c)$ is a dimensionless diffusion coefficient (independent of Σ_t). In a purely absorbing medium ($c = 0$), the diffusion coefficient is $D_0(c=0) = 1$. In a scattering-dominated medium ($c = 1$), $D_0(c=1) = \frac{1}{3}$, while between those two limits, $D_0(c)$ varies monotonically (see Fig. 2a below in Sec. III.F). In problems that have $c > 1$ (e.g., multiplying media), we set $k_0 \equiv i\varkappa_0$, and the main conclusions remain the same. In Ref. 8 there is an extensive set of tabulated values of $\varkappa_0(c)$ or $D_0(c)$, in addition to approximate expressions for these parameters. We quote some of them that will be of use later in this paper. For $c \ll 1$ (highly absorbing media), one can write

$$D_0(c) \approx (1-c)\left[1 + 4e^{-2/c} + 8\frac{c+2}{c}e^{-4/c} + 12\frac{8+4c+c^2}{c^2}e^{-6/c} + 16\frac{128+48c+18c^2+3c^3}{3c^3}e^{-8/c} + \cdots\right] , \quad (22)$$

while for $(1-c) \ll 1$ (highly scattering media),

$$D_0(c) \approx \frac{1}{3}\left[1+\frac{4}{5}(1-c)+\frac{108}{175}(1-c)^2+\frac{396}{875}(1-c)^3+\frac{828}{2695}(1-c)^4+\frac{568,836}{3,128,125}(1-c)^5+\cdots\right] . \tag{23}$$

In Ref. 10 there is an approximate expression for $D_0(c)$ with a maximal error of 1%:

$$D_0(c) = \begin{cases} (1-c) & c \le 0.3 \\ \dfrac{4}{\pi^2 c}\left(\dfrac{c-0.0854}{c+0.112}\right) & c > 0.3 \end{cases} . \tag{24}$$

One can write a better approximation of $D_0(c)$ with a maximal error of 0.3% (Ref. 35[d]):

$$D_0(c) = \frac{0.3267567 + c[0.1587312 - c(0.5665676 + c)]}{0.1326495 + c[0.03424169 + c(0.1774006 - c)]} \cdot \left(\frac{0.40528473}{1+c}\right) . \tag{25}$$

The modified Fick's law [Eq. (20)] along with the conservation law [Eq. (2)] provides a diffusion equation that is presumed to predict the asymptotic scalar flux.

To examine the capabilities and the limitations of the asymptotic diffusion theory, we introduce the exact Boltzmann solution of the plane source problem $[Q(x) = Q_0\delta(x)]$ in one-dimensional slab geometry. As explained, the solution may be expressed as a composition of two parts: an asymptotic part, which is valid far from sources and boundaries between media, and a transient part, which is valid especially near sources and boundaries and decays relatively fast compared to the asymptotic part[7,8]:

$$\phi_{exact} = \underbrace{\frac{Q_0}{2}\frac{dk_0^2(c)}{dc}\frac{1}{\varkappa_0(c)}e^{-\varkappa_0(c)\Sigma_t|x|}}_{Asymptotic\text{–}Part} + \underbrace{\frac{Q_0}{2}\int_0^1 \frac{g(c,\mu)}{\mu}e^{-\Sigma_t(|x|/\mu)}\,d\mu}_{Transient\text{–}Part} , \tag{26}$$

where

$$k_0 = i\varkappa_0$$

and

$$g(c,\mu) = [(1-c\mu\tanh^{-1}\mu)^2 + ((\pi/2)c\mu)^2]^{-1} .$$

[d] Fitted curve of $D_0(c)$ to the tabulated values introduced in Ref. 8.

A common convenient definition is $1/N_{0+} \equiv (dk_0^2(c)/dc)(1/\varkappa_0(c))$. If we check the ratio $\phi_{as}(\Sigma_t x)/\phi(\Sigma_t x)$, we will see that for a purely scattering medium ($c = 1$), the asymptotic solution coincides with the exact solution; i.e., $\phi_{as}(\Sigma_t x)/\phi(\Sigma_t x) = 1$ everywhere. The percentile of the asymptotic part decreases with c. For example, for $c = 0.5$, we obtained $\phi_{as}(\Sigma_t x)/\phi(\Sigma_t x) \approx 50\%$ at 1 mean free path from the source. For a purely absorbing medium (in the isotropic plane source problem), $\phi_{as}(\Sigma_t x)/\phi(\Sigma_t x) = 0$ everywhere; the asymptotic part in Eq. (26) is zero $[(dk_0^2(c)/dc)|_{c=0} = 0]$, and the transient (exact) solution can be expressed after simple transformation as $[g(c=0,\mu) = 1]$:

$$\phi_{exact}(x) = \frac{Q_0}{2}E_1(\Sigma_t x) = \phi_{transient}(x) , \tag{27}$$

where $E_1(x)$ is called the exponential integral and is defined by $E_1(x) = \int_{|x|}^{\infty}(e^{-x'}/x')\,dx'$.

As was said before, the asymptotic diffusion approximation is presumed to predict only the asymptotic part of the exact solution. Solving the asymptotic diffusion equation with a plane source gives this solution[1,2]:

$$\phi_{AD}(x) = \frac{Q_0}{2}\frac{1}{\varkappa_0(c)D_0(c)}e^{-\varkappa_0(c)\Sigma_t|x|} = \frac{Q_0}{2}\frac{\varkappa_0(c)}{1-c}e^{-\varkappa_0(c)\Sigma_t|x|} . \tag{28}$$

For the purely absorbing case $[c = 0$, when $\phi_{as}(\Sigma_a x)/\phi(\Sigma_a x) = 0]$, Eq. (28) yields

$$\phi_{AD}(x) = \frac{Q_0}{2}e^{-\Sigma_a|x|} , \tag{29}$$

which is very different from the exact solution of the isotropic plane source problem [Eq. (27)] as explained. For the highly scattering case $[(1-c) \ll 1]$, Eq. (28) yields

$$\phi_{AD}(x) = \frac{Q_0}{2}\sqrt{\frac{3}{1-c}}\cdot e^{-\sqrt{3(1-c)}\Sigma_t|x|} , \tag{30}$$

which is identical to both the classical diffusion solution, Eq. (12), and the exact solution of the plane source problem, Eq. (26), for $(1-c) \ll 1$.

The solution of the asymptotic diffusion equation [Eq. (28)] restores the exact exponent decay rate of the asymptotic solution for any given c. On the other hand, the asymptotic diffusion solution [Eq. (28)] has only partial success in restoring the amplitude of the asymptotic part of the exact solution [Eq. (26)]. For highly scattering media ($c \approx 1$), the two solutions merge, and the error in the amplitude of the asymptotic diffusion solution increases as c decreases. This error is added to the fact that the amplitude of the asymptotic part of the exact solution is small anyway when c decreases, as mentioned before. The main conclusion is that the asymptotic diffusion approximation is accurate for highly scattering media, and the accuracy of this approximation decreases with c (although it is still much better than the classic diffusion approximation). Reference 8 is focused on an extensive discussion on the comparison between the asymptotic solutions, the transient solutions, and the diffusion solutions.

The derivation of the asymptotic diffusion approximation was introduced here in a one-dimensional slab geometry. The generalization of the analysis to a multidimensional general geometry is straightforward and well known. Roughly, to formalize the general Fick's law, one can replace $\partial/\partial x$ in Eq. (20) with ∇. For a full development of the modified Fick's law in general geometry (derived from the exact integral Boltzmann equation), see Chapter IV of Ref. 8 or Chapter III(3) of Ref. 4. Equation (20) along with the conservation law [Eq. (2)] defines the asymptotic diffusion approximation. The asymptotic diffusion approximation can be easily extended to include nonhomogeneous media,[4,36] to a multigroup case,[10] and to linear anisotropic media.[37,38]

III.B. The Basic Rationale

Having introduced the asymptotic diffusion approximation, we can introduce our analogous approximation to time-dependent problems. This approximation allows us to abandon the flux limiters[10–15] or the Eddington factor approximations[13,15] (which usually contain nonlinear equations, due to the flux-dependent diffusion coefficient or Eddington factor).

As was explained earlier, examining the form of the solution of the Telegrapher's equation [Eq. (13) or Eq. (14)] shows that the magnitude of error in the steady-state eigenvalue is *the same* magnitude of error in the particle velocity (both include the $\sqrt{3}$ term). The appearance of the factor of 3 is caused by the approximation for the second moment of the angular flux, Eq. (6), which contains a factor of 3. This approximation yields the form of Eq. (7).

Equation (7) contains two terms that include a factor of 3: The first term is $3\Sigma_t(\vec{r})\vec{J}(\vec{r},t)$. Remembering the definition of the diffusion coefficient, $D = 1/3\Sigma_t$ [Eq. (8)], this term can be written as $(1/D)\vec{J}(\vec{r},t)$. Using the asymptotic diffusion approximation, we fix D with the medium-dependent definition in Eq. (21) (this sets the correct steady-state eigenvalue). The second term containing a factor of 3 is the $(3/v)(\partial\vec{J}(\vec{r},t)/\partial t)$ term, from which is obvious that this term is responsible to the error in the particle velocity; this term contains a $v/3$ factor.

The main rationale behind our new approximation is as follows. First, the P_1 equations (or alternatively, the Telegrapher's equation) are inherently flux-limited equations, but with the wrong velocity (i.e., over flux-limited). Second, the P_1 equations contain two equations: an exact equation, the conservation law, and an approximate equation, a time-dependent Fick's law, which contains the terms that include the factor of 3. The rationale says that we must not change the exact equation, while we are free to develop a modified time-dependent Fick's law that will replace the (wrongly) approximate equation. This rationale follows the procedure of the asymptotic diffusion approximation in the time-independent steady-state case: developing a modified Fick's law [Eq. (20)] instead of the classic Fick's law. This rationale urges us to find modified P_1 equations of this form (in the one-dimensional slab geometry case):

$$\frac{1}{v}\frac{\partial\phi(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} + \Sigma_a\phi(x,t) = Q(x) \tag{31a}$$

and

$$\frac{\mathcal{A}}{v}\frac{\partial J(x,t)}{\partial t} + \frac{\partial\phi(x,t)}{\partial x} + \mathcal{B}\Sigma_t J(x,t) = 0 \ . \tag{31b}$$

Equation (31a) is the conservation law, while Eq. (31b) is a modified equation in which we replaced the factor of 3 in Eq. (7) by two (albedo-dependent) different parameters: $\mathcal{A}$ and $\mathcal{B}$ [although the factor of 3 has the same origin, Eq. (6)]. $\mathcal{B}$ will account for the steady-state solution as in the asymptotic diffusion equation [actually, it should be exactly $1/D$ because Eq. (31b) should reduce to Eq. (20) in steady state]. $\mathcal{A}$ should account for the time behavior, particularly the particle velocity. Dividing the treatment of the steady-state term and the time evolution term into two albedo-dependent parts reproduces the correct asymptotic behavior in time as well as in space.

With such a separation, we can derive the Telegrapher's equation using the same procedure as the classic Telegrapher's equation, using the modified P_1 equations, Eqs. (31), instead of the classical P_1 equations, Eqs. (2) and (7):

$$\begin{aligned}\left[\frac{\mathcal{A}}{v}\right]\frac{\partial^2\phi(x,t)}{\partial t^2} - [v]\frac{\partial^2\phi(x,t)}{\partial x^2}& \\ + [\mathcal{A}\Sigma_a + \mathcal{B}\Sigma_t]\frac{\partial\phi(x,t)}{\partial t} &+ [v\mathcal{B}\Sigma_a\Sigma_t]\phi(x,t) \\ = v\mathcal{B}\Sigma_t Q(x) \ .&\end{aligned} \tag{32}$$

We solve this modified Telegrapher's equation for the problem of a delta source, as introduced in Sec. II, for a general medium (using the Fourier domain on space and Laplace domain on time). The scalar flux [analogous to the solution of the classical Telegrapher's equation, Eq. (13)] is

$$\phi(x,t) = \frac{\mathcal{B}\Sigma_t v Q_0}{2\sqrt{\mathcal{A}}}\int_0^t e^{-(v/2)(\Sigma_a+(\mathcal{B}/\mathcal{A})\Sigma_t)u} \cdot I_0\left[\frac{v}{2}\left(\Sigma_a - \frac{\mathcal{B}}{\mathcal{A}}\Sigma_t\right)\sqrt{u^2 - \mathcal{A}\frac{|x|^2}{v^2}}\right] \cdot H\left(u - \sqrt{\mathcal{A}}\frac{|x|}{v}\right) du \quad . \tag{33}$$

We can easily see the satisfactory results. The particle velocity depends only on $\mathcal{A}$: $v/\sqrt{\mathcal{A}}$. As far as the particle velocity is concerned, we would always require that $\mathcal{A} = 1$ for any albedo (medium). But, we notice that $\mathcal{A}$ also appears in other time-dependent terms, such as the time exponent decay or inside the modified Bessel term; there, $\mathcal{A} = 1$ may not always be the preferable choice for any medium.

Taking $t \to \infty$ [using Eq. (34)] yields the steady-state behavior:

$$\phi(x, t\to\infty) = \frac{Q_0}{2}\sqrt{\frac{\mathcal{B}\Sigma_t}{\Sigma_a}}e^{-\sqrt{\mathcal{B}\Sigma_t\Sigma_a}|x|} \quad , \tag{34}$$

which depends only on $\mathcal{B}$. Both Eqs. (33) and (34) reduce to the classical Telegrapher's equation solutions, when $\mathcal{A} = \mathcal{B} = 3$.

So far, we have explained the benefit of separating the treatment of the time evolution and the steady-state solution. Now, we have to develop a modified time-dependent Fick's law from the exact time-dependent Boltzmann equation, exactly as Case et al. did with the time-independent case.

Developing a modified time-dependent Fick's law in the *x-t* domain would force us to choose an intelligent trial function for the asymptotic flux in time as well as in space. Examples of works that follow this approach are Refs. 25, 39, and 40, but this method is not exact and only approximates the asymptotic flux. Actually, the exact time-dependent asymptotic solution is not simple at all (compared to the time-independent solution), as can be seen in the classical textbooks such as Ref. 7. To obtain the accurate time-asymptotic behavior from the exact time-dependent Boltzmann equation, we use the following maneuver. Applying a Laplace transform on time over Eq. (31b),

$$\hat{J}_s(x) = -\frac{v}{\mathcal{A}s + \mathcal{B}v\Sigma_t}\cdot\frac{\partial\hat{\phi}_s(x)}{\partial x} \quad . \tag{35}$$

Equation (35) is in the *x-s* domain and has the form of a Fick's law with a diffusion coefficient:

$$\hat{D}_s(\mathcal{A},\mathcal{B}) \equiv v\,\frac{1}{\mathcal{A}s + \mathcal{B}v\Sigma_t} \quad . \tag{36}$$

The procedure is now straightforward. We should follow the prescription for solving the time-dependent Boltzmann equation using the Laplace domain on time as shown in Refs. 7 and 8 and obtain a modified (albedo and *s*-dependent) diffusion coefficient, solving for $\mathcal{A}$ and $\mathcal{B}$.

III.C. The Time-Dependent Fick's Law

In this section, we derive a modified time-dependent Fick's law from the exact time-dependent monoenergetic Boltzmann equation.[7] We introduce the derivation in one-dimensional slab geometry, but the expansion to a multidimensional general geometry is simple and straightforward, as described in Chapter IV of Ref. 8 or Chapter III(3) of Ref. 4. The time-dependent monoenergetic Boltzmann equation in a slab geometry is

$$\frac{1}{v}\frac{\partial\psi(x,\mu,t)}{\partial t} + \mu\frac{\partial\psi(x,\mu,t)}{\partial x} + \Sigma_t\psi(x,\mu,t) = \frac{c\Sigma_t}{2}\int_{-1}^{1}\psi(x,\mu',t)\,d\mu' \quad . \tag{37}$$

Applying the Laplace transform on time in Eq. (37) yields

$$\left(\frac{s}{v} + \Sigma_t\right)\hat{\psi}_s(x,\mu) + \mu\frac{\partial\hat{\psi}_s(x,\mu)}{\partial x} = \frac{c\Sigma_t}{2}\int_{-1}^{1}\hat{\psi}_s(x,\mu')\,d\mu' \quad . \tag{38}$$

We define a modified total cross section $\hat{\Sigma}_t^s$ and an albedo $\hat{c}_s$, both s dependent[8]:

$$\hat{\Sigma}_t^s \equiv \Sigma_t + \frac{s}{v} \tag{39a}$$

and

$$\hat{c}_s \equiv \frac{\Sigma_s}{\hat{\Sigma}_t^s} = \frac{c}{1 + \dfrac{s}{v\Sigma_t}} \quad . \tag{39b}$$

Substituting the definition of Eqs. (39) in Eq. (38) gives

$$\mu\frac{\partial\hat{\psi}_s(x,\mu)}{\partial x} + \hat{\Sigma}_t^s\hat{\psi}_s(x,\mu) = \frac{\hat{c}_s\hat{\Sigma}_t^s}{2}\int_{-1}^{1}\hat{\psi}_s(x,\mu')\,d\mu' \quad . \tag{40}$$

Equation (40) has the form of the time-independent Boltzmann equation, used to derive the asymptotic diffusion approximation, Eq. (16). Henceforth, the procedure is straightforward, and we will describe it shortly, following the procedure described in Sec. III.A. Assuming the separation of variables as before, we obtain the s-dependent eigenvalue $\hat{\varkappa}_{s,0}$:

$$\frac{2}{\hat{c}_s} = \frac{1}{\hat{\varkappa}_{s,0}} \ln\left(\frac{1+\hat{\varkappa}_{s,0}}{1-\hat{\varkappa}_{s,0}}\right) . \tag{41}$$

The time-dependent asymptotic angular flux (asymptotic only in space and exact in time) is

$$\hat{\psi}_s(x,\mu) = \frac{\hat{c}_s}{2} A_0 \frac{e^{\hat{\varkappa}_{s,0}\hat{\Sigma}_t^s x}}{1+\mu\hat{\varkappa}_{s,0}} + \frac{\hat{c}_s}{2} B_0 \frac{e^{-\hat{\varkappa}_{s,0}\hat{\Sigma}_t^s x}}{1-\mu\hat{\varkappa}_{s,0}} . \tag{42}$$

The first two moments of the angular flux are

$$\hat{\phi}_s(x) = A_0 e^{\hat{\varkappa}_{s,0}\hat{\Sigma}_t^s x} + B_0 e^{-\hat{\varkappa}_{s,0}\hat{\Sigma}_t^s x} \tag{43a}$$

and

$$\hat{J}_s(x) = \left(\frac{\hat{c}_s - 1}{\hat{\varkappa}_{s,0}}\right)\left[A_0 e^{\hat{\varkappa}_{s,0}\hat{\Sigma}_t^s x} - B_0 e^{-\hat{\varkappa}_{s,0}\hat{\Sigma}_t^s x}\right] . \tag{43b}$$

Recognizing that the connection between the first two moments yields a Fick's law of this form (exactly as in the time-independent case),

$$\hat{J}_s(x) = -\hat{D}_s(\hat{c}_s, \hat{\Sigma}_t^s) \cdot \frac{\partial \hat{\phi}_s(x)}{\partial x} , \tag{44}$$

with an s-dependent diffusion coefficient

$$\hat{D}_s(\hat{c}_s, \hat{\Sigma}_t^s) \equiv \frac{1-\hat{c}_s}{\hat{\varkappa}_{s,0}^2 \hat{\Sigma}_t^s} . \tag{45}$$

Substituting the definition of the modified total cross section and albedo [Eqs. (39)] in Eq. (45) gives this diffusion coefficient (in the Laplace domain):

$$\hat{D}_s(\hat{c}_s, \Sigma_t) = v \frac{v\Sigma_t(1-c)+s}{(v\Sigma_t+s)^2 \hat{\varkappa}_{s,0}^2} \equiv v \frac{D_0(\hat{c}_s)}{v\Sigma_t + s} . \tag{46}$$

The diffusion coefficient is s dependent, and the s dependency exists also in $\hat{\varkappa}_{s,0}$ or, alternatively, in $D_0(\hat{c}_s)$. There are transcendental closed equations for both $\hat{\varkappa}_{s,0}$ and $D_0(\hat{c}_s)$, and we can use the approximate expressions in Ref. 8; some of them were introduced earlier in Sec. III.A.

A similar approximation may be developed from the work of Bengston.[41] In this work a new asymptotic diffusion is developed to include time-dependent problems (this work remains in a diffusion-type equation). This work assumes that the flux time dependency has an exponential form $-\psi(x,\mu,t) = \psi_{\alpha_0}(x,\mu)e^{\alpha_0 t}$, when α_0 is the most dominant time eigenvalue. This assumption is quite reasonable in the asymptotic time regime (sufficiently large times) as shown in Chapter 3.2 of Ref. 5. This choice leads to equations similar to these shown in this section (replacing s with α_0). Assuming this, one obtains a diffusion coefficient similar to Eq. (46), replacing s with α_0. Bengston's main conclusion is that the asymptotic diffusion approximation is preferable to the *classic* Telegrapher's equation approximation (using $\mathcal{A} = \mathcal{B} = 3$).

Nevertheless, this work has no analogy to modified P_1 equations (or the modified Telegrapher's equation). Keeping the form of a diffusion-like equation that contains a diffusion coefficient that depends on α_0 requires an intelligent guess for α_0. In addition, keeping our rationale (of developing new modified P_1 equations) using α_0 has disadvantages too; this approach is useful only when $|\alpha_0| \to 0$, i.e., only in highly (infinitely) scattering media ($c \approx 1$) (or in critical systems in neutronics), as shown in Ref. 41.

So far, the derivation has been exact (asymptotic on space, of course). We should now take Eq. (46) and compare it to Eq. (36) to solve for $\mathcal{A}$ and $\mathcal{B}$ for a given albedo c, using one of the definitions of $D_0(\hat{c}_s)$ [or, alternatively, of $\hat{\varkappa}_{s,0}$; for convenience, we work with $D_0(\hat{c}_s)$].

III.D. Purely Absorbing Media

For a purely absorbing medium, we can solve for $\mathcal{A}$ and $\mathcal{B}$ exactly. Since in a purely absorbing medium, $\hat{c}_s = c = 0$, the expression for $\hat{\varkappa}_{s,0}(0)$ or $D_0(\hat{c}_s)$ is s independent and equal to $\hat{\varkappa}_{s,0}(0) = \varkappa_0(0) = D_0(0) = 1$. In this case, the diffusion coefficient [Eq. (46)] can be written as

$$\hat{D}_s(c=0, \Sigma_t) = v \frac{1}{(v\Sigma_t + s)} . \tag{47}$$

Comparing Eq. (47) to Eq. (36) allows us to solve for $\mathcal{A}$ and $\mathcal{B}$ for the purely absorbing medium: $\mathcal{B} = \mathcal{A} = 1$. This result is accurate since $\mathcal{B} = 1/D_0 = 1$ reproduces the asymptotic steady-state behavior for the purely absorbing case and $\mathcal{A} = 1$ gives the correct particle velocity.

For example, the asymptotic Telegrapher's equation solution for the time-dependent problem for the purely absorbing case with a constant plane source [$Q(x) = Q_0\delta(x)$], using Eq. (33), setting $\mathcal{B} = \mathcal{A} = 1$, and solving the integral gives

$$\phi(x,t) = \frac{Q_0}{2}(e^{-\Sigma_a|x|} - e^{-v\Sigma_a t})H\left(t - \frac{|x|}{v}\right) . \tag{48}$$

The solution of Eq. (48) is similar to the classical Telegrapher's equation solution [Eq. (14)] except for the disappearance of the $\sqrt{3}$ factor from the the amplitude, the decay rate, and the particle velocity (the correct particle velocity). Comparison to the exact solution is done below.

III.E. Scattering-Dominated Media

For any other case except the purely absorbing case, we need to force an asymptotic treatment on the time domain, as well as the asymptotic derivation in the space domain. As mentioned earlier, the asymptotic diffusion approximation should describe the asymptotic regime on space, i.e., far from sources or interface between two different media. The equivalent terminology for time-asymptotic behavior should be far enough from $t = 0$, i.e., as $t \to \infty$. For this reason we use the Final Value Theorem, which helps us define the asymptotic regime in the s domain (instead of the t domain):

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}(s\hat{F}(s)) \ . \tag{49}$$

Since we take $s \to 0$, we can approximate $D_0(\hat{c}_s)$ for the highly scattering case as $D_0(\hat{c}_s \approx 1) \approx \frac{1}{3}$, since the correction for $\hat{c}_s$ is negligible in comparison to c. This choice will lead to this diffusion coefficient [Eq. (46)]:

$$\hat{D}_s(c = 1, \Sigma_t) = v\,\frac{1}{3(v\Sigma_t + s)} \ . \tag{50}$$

Solving for $\mathcal{A}$ and $\mathcal{B}$ gives us $\mathcal{B} = \mathcal{A} = 1/D_0(c = 1) = 3$. As far as the steady-state behavior is concerned, this choice is good, but as far as the time behavior is concerned, this choice deviates at the exact same magnitude as the classical Telegrapher's equation (P_1) approximation does.

Next, we evaluate $D_0(\hat{c}_s)$ more carefully (by not neglecting the s dependency) using the approximate Eq. (23): $D_0(\hat{c}_s \approx 1) \approx \frac{1}{3}(1 + \frac{4}{5}(1 - \hat{c}_s) + \cdots)$ and substituting it in Eq. (46). This expression is complex, and we cannot solve for $\mathcal{A}$ and $\mathcal{B}$ explicitly. Since we look for the asymptotic behavior in time ($s \to 0$), we expand the inverse of the diffusion coefficient in Eq. (46) in a Taylor series:

$$\frac{v}{\hat{D}_s(c, \Sigma_t)} = \frac{v\Sigma_t + s}{D_0(\hat{c}_s)} \approx \mathcal{B}v\Sigma_t + \mathcal{A}s + \mathcal{O}(s^2) + \cdots \ . \tag{51}$$

Solving for $\mathcal{A}$ and $\mathcal{B}$ for the highly scattering medium is easy: $\mathcal{B} = 3$ and $\mathcal{A} = \frac{3}{5}$. The steady-state behavior is accurate ($\mathcal{B} = 3$), and the magnitude of the error of the particle velocity decreases to $v/\sqrt{\mathcal{A}} = \sqrt{\frac{5}{3}}v \approx 1.3v$, which is smaller than the classical P_1 prediction. For charged particles in the isotropic case, Refs. 26 and 27 share similar conclusions based on asymptotic analysis (depend on the scattering-isotropical model, which yields particle velocity that equals $\sqrt{\frac{5}{3}}v$ or $\sqrt{\frac{5}{11}}v$).

Again, for example, the asymptotic Telegrapher's equation solution for the time-dependent problem for the purely scattering case with a constant plane source, using Eq. (33), setting $\mathcal{B} = 3$, and $\mathcal{A} = \frac{3}{5}$ gives

$$\begin{aligned}\phi(x,t) = \frac{\sqrt{15}\Sigma_t v Q_0}{2}\int_0^t & e^{-5(v/2)\Sigma_t u} \\ & \cdot I_0\left[5\,\frac{v}{2}\,\Sigma_t\sqrt{u^2 - \frac{3}{5}\,\frac{|x|^2}{v^2}}\,\right] \\ & \cdot H\left(u - \sqrt{\frac{3}{5}}\,\frac{|x|}{v}\right) du \ . \end{aligned} \tag{52}$$

We see that the particle advances at a velocity $\sqrt{\frac{5}{3}}v$. Comparison to the exact solution is shown below.

Bengston[41] developed a diffusion coefficient that is similar to Eq. (51), replacing s with α_0, and determined that the diffusion coefficient behaves like

$$\frac{1}{D} \approx 3\Sigma_t\left(\frac{1 + 4c}{5} + \frac{1}{5}\,\frac{\alpha_0}{\Sigma_t v} + \cdots\right) \ . \tag{53}$$

As was said before, Ref. 41 kept the form of the diffusion equation, and thus, one has to guess the correct α_0 to use Eq. (53). Following our rationale (of finding modified coefficients for the P_1 equations) using α_0 instead of the full s-domain treatment is possible only if $|\alpha_0| \to 0$. Otherwise, we cannot expand $D_0(c, \alpha_0)$ in a Taylor series in order to solve for $\mathcal{A}$ and $\mathcal{B}$. But, $|\alpha_0| \to 0$ only for a scattering-dominated (infinite) medium [$c \approx 1$ as shown in Eq. (53)] or in critical systems in neutronics, and not for a general medium as required. In a general medium, and especially in an absorbing-dominated medium ($c \approx 0$), even if we can use the approximation of single time eigenvalue, this single eigenvalue is not necessarily small.

III.F. General Media

For a general medium (general c), we can use one of the approximations for $D_0(\hat{c}_s)$ that was introduced in Sec. III.A, or, alternatively, for $\hat{\varkappa}_{s,0}$. We repeat the process that was introduced for the highly scattering case by expanding the diffusion coefficient in a Taylor series, solving for $\mathcal{A}$ and $\mathcal{B}$ for a general medium. As expected, $\mathcal{B}(c) = 1/D_0(c)$ exactly (see Fig. 2a), while $\mathcal{A}(c)$ is shown in Fig. 2b for several approximations. For the most accurate analytical expression for $D_0(c)$ [Eq. (25)], we introduce an analytical approximation for $\mathcal{A}(c)$ for a general medium:

$$\mathcal{A}(c) = \frac{0.247(0.433 + 0.421c - 2.681c^2 - 1.82c^3 + 4.9c^4 - 1.058c^5 + 2.56c^6)}{(0.327 + 0.159c - 0.567c^2 - c^3)^2} \ . \tag{54}$$

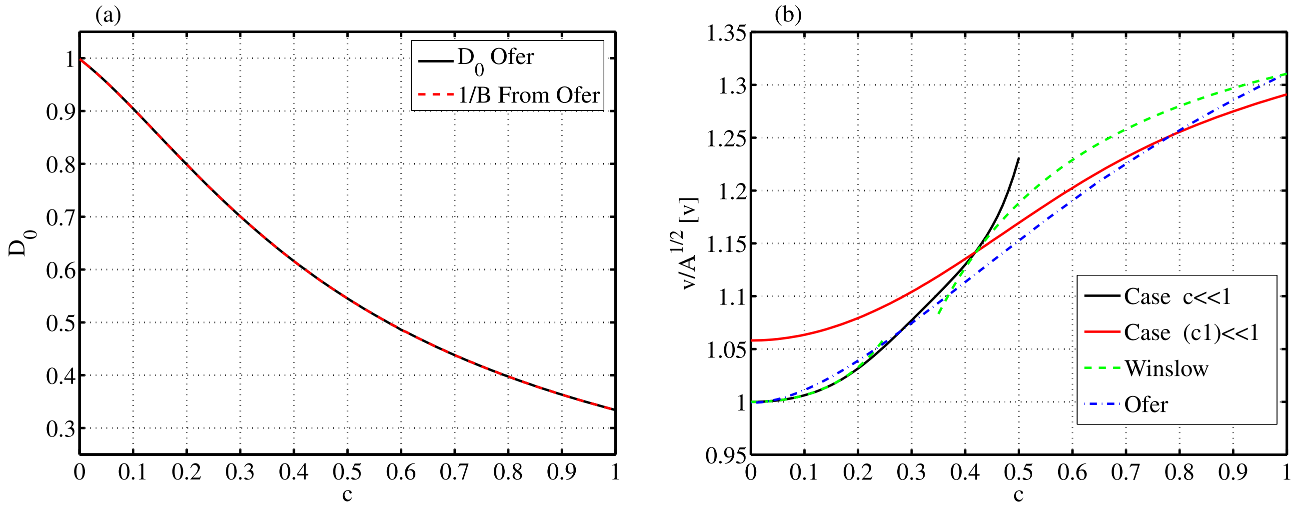

Fig. 2. (a) Comparison between the dimensionless diffusion coefficient $D_0(c)$ and $1/\mathcal{B}$ calculated from the analysis in Sec. III.F using Ofer's formula. (b) The particle velocity $(v/\sqrt{\mathcal{A}})$ as a function of albedo c, calculated from the analysis described at Sec. III.F using different approximations for $D_0(c)$.

The result is satisfying: The steady-state behavior is reproduced exactly, and the time behavior is described quite well. The error in the particle velocity is $\sim$30% when $c = 1$ (scattering media), and it deceases monotonically to zero when $c = 0$ (purely absorbing media; see Fig. 2b). We note here that the main conclusions remain the same for the problems with $c > 1$ such as neutronics. The error in the particle velocity when $c > 1$ is a little bit larger but is always $<1.45v$ [when $c > 1$, we recommend using Eq. (24) instead of Eq. (25) as an approximation for $D_0(c)$].

Since the procedure introduced here is asymptotic both in space and time, we call this approximation "the asymptotic Telegrapher's equation (P_1) approximation."

IV. COMPARISON WITH EXACT SOLUTIONS

To check the capabilities of the new approximation (other than the success in getting a more accurate particle velocity), we want to compare the solution of the scalar flux resulting from the new approximation versus the exact transport solutions. An analytic solution for the Boltzmann equation is available only for a few limiting cases. Even for the steady-state case, an analytic solution is available only for a purely absorbing medium while for a general c the solution involves a transcendental equation for $\varkappa_0(c)$. First, in Sec. IV.A, we compare our solution to the time-dependent problem with a time-independent plane source $Q(x) = Q_0\delta(x)$. Next, in Sec. IV.B, we compare our approximation with semianalytical published solutions for a time-dependent plane source $Q(x,t) = Q_0\delta(x)\delta(t)$ (the AZURV1 benchmark[42]).

IV.A. Time-Independent Plane Source

In Sec. III.C we have shown that it is possible to transform the time-dependent equation [Eq. (37)] to a time-independent equation [Eq. (40)] by applying the Laplace transform and using an effective cross section and albedo as introduced in Eqs. (39). The equation to be solved is identical to Eq. (37), only it has a source term. Since the source in our problem is time independent, the source term should be multiplied by s, i.e., $sQ_0\delta(x)$ (the Laplace transform of 1 is s). Thus, to obtain the time-dependent solution, we need to apply the inverse Laplace transform to the time-independent solution (which is well known) under the transformation $Q_0 \to Q_0/s$ and, of course, $\Sigma_t \to \hat{\Sigma}_t(s)$ and $c \to \hat{c}(s)$.

We start with the purely absorbing case [$c\hat{(s)} = c = 0$]. The exact time-independent solution in the purely absorbing case appears in Eq. (27). Applying the transformation mentioned above, the solution for the time-dependent problem in the s domain is

$$\hat{\phi}_{exact}(x,s) = \frac{Q_0}{2s} E_1\left[\frac{|x|}{v}(v\Sigma_a + s)\right] . \quad (55)$$

Applying the inverse Laplace transform to Eq. (55) gives the exact time-dependent solution for the purely absorbing case[31e]:

$$\phi_{exact}(x,t) = \frac{Q_0}{2}[E_1(|x|\Sigma_a) - E_1(v\Sigma_a t)]H\left(t - \frac{|x|}{v}\right) . \quad (56)$$

[e]For the exponential-integral we use the pair on p. 322, No 19.1.1, of Ref. 31.

The exact solution [Eq. (56)] is very different from the asymptotic Telegrapher's equation solution [Eq. (48)]. This is not surprising since even in the steady-state case (time-independent equation), the asymptotic diffusion equation solution is very different from the exact transport solution in the isotropic plane source problem at $c = 0$ (as discussed comprehensively in Sec. III.A). But, we point out an interesting curiosity: The time behavior of the two solutions is qualitatively similar: The exact steady-state solution [Eq. (27)] has the form of an exponential integral while the asymptotic (steady-state) solution [Eq. (29)] has the form of a decaying exponential. The time-dependent exact solution [Eq. (56)] has the form of the difference of two exponential integrals multiplied by the Heaviside step function, and the time-dependent asymptotic Telegrapher's equation [Eq. (48)] has the form of the difference of exponents multiplied by the Heaviside function.

Since at $c = 0$ the asymptotic solution is not supposed to predict the exact solution (in the isotropic plane source problem), we present the interesting extreme case $(1 - c) \ll 1$, the highly scattering case. In that case, the asymptotic description is supposed to predict the exact solution correctly (at $c = 1$ in the steady-state case, the ratio $\phi_{as}(\Sigma_t x)/\phi(\Sigma_t x) = 1$ exactly).

In the time-dependent case, $\hat{c}(s) \equiv \Sigma_s/\hat{\Sigma}_t^s = c/(1 + s/v\Sigma_t)$ replaces c. Since we are interested in the asymptotic behavior in time (as well as in space), we assume that s is small, and therefore, $(1 - \hat{c}(s)) \ll 1$. Taking the asymptotic part of the steady-state solution [Eq. (26)], applying the transformations mentioned above to the time-dependent problem in the s domain, and substituting the expressions of the transcendental expressions for $(1 - c) \ll 1$, $dk_0^2(c)/dc \approx 3(1 - \frac{8}{5}(1 - c))$ and $\varkappa_0(c) \approx \sqrt{3(1 - c)(1 - \frac{4}{5}(1 - c))}$, give the scalar flux for the time-dependent case (in the s domain):

$$\hat{\phi}_{asymptotic}(x, s) = \frac{Q_0}{2}\sqrt{\frac{3}{5}}\,\frac{5v\Sigma_t - 3s}{s\sqrt{s(s + 5v\Sigma_t)}}\,e^{-\sqrt{3/5}|x/v|\sqrt{s(s+5v\Sigma_t)}} \quad . \tag{57}$$

We note that there are several approximations to the transcendental expressions for $dk_0^2(c)/dc$ and $\varkappa_0(c)$. We took the most convenient ones to apply the inverse Laplace transform easily. Applying the inverse Laplace transform to Eq. (57) gives the asymptotic time-dependent solution for large t for the purely scattering case:

$$\begin{aligned}\phi(x, t) = {} & \frac{\sqrt{15}\Sigma_t v Q_0}{2}\int_0^t e^{-5(v/2)\Sigma_t u}\cdot I_0\left[5\,\frac{v}{2}\,\Sigma_t\sqrt{u^2 - \frac{3}{5}\,\frac{|x|^2}{v^2}}\right]\cdot H\left(u - \sqrt{\frac{3}{5}}\,\frac{|x|}{v}\right)du \\ & - \frac{Q_0}{2}\sqrt{\frac{27}{5}}\,e^{-5(v/2)\Sigma_t t}\cdot I_0\left[5\,\frac{v}{2}\,\Sigma_t\sqrt{t^2 - \frac{3}{5}\,\frac{|x|^2}{v^2}}\right]\cdot H\left(t - \sqrt{\frac{3}{5}}\,\frac{|x|}{v}\right) \quad . \end{aligned} \tag{58}$$

The first term in Eq. (58) is exactly the solution of the asymptotic Telegrapher's equation approximation in the purely scattering case [Eq. (52)]. The second term of Eq. (58) is a nonphysical term, causing a negative flux at early times, and it results from the approximations to the transcendental expressions, while changing the approximations changes the magnitude of this term. Anyhow, this term is negligible at long times, and we will ignore it. Here, in the scattering-dominated regime the asymptotic Telegrapher's equation predicts the correct asymptotic solution, in addition to the particle velocity, and gives strong validation for our approximation. (The particle velocity in the exact solution is not exactly v since we took the approximate expressions for the transcendental functions, assuming small s. Taking the exact expressions for the transcendental functions including the $\mathcal{O}(1 - c)^2$ terms and so on will give the correct particle velocity, but then an analytic inverse Laplace transform is unavailable.)

Using the modified coefficients $\mathcal{A}(c)$ and $\mathcal{B}(c)$, the scalar flux of the one-dimensional time-independent source problem is calculated for different values of c (the same values of Fig. 1). In Fig. 3 the scalar flux resulting from the asymptotic Telegrapher's equation solution is shown [Eq. (33)] along with the asymptotic diffusion solution [Eq. (11), using $D_0(c)$ instead of a constant diffusion coefficient (dashed lines)]. In addition we plot the Boltzmann solution of two of the limiting cases.

We see that in early times, the particle front is advancing with a velocity that is close to the real particle velocity v (in $c = 0$ the propagating speed is exact). At longer times the solution for the asymptotic Telegrapher's equation decays to the solution of the asymptotic diffusion solution. As expected and explained, at $c = 0$ the exact solution is far from both the asymptotic diffusion and the asymptotic Telegrapher's equation (except the wave front). At $c \approx 1$ the asymptotic Telegrapher's equation yields the asymptotic Boltzmann solution for large t. It is important to note that as far as the particle velocity is concerned, the preferable choice for $\mathcal{A}$ is $\mathcal{A}(c) = 1$ for any given c. But, in contrast to $\mathcal{B}(c)$, which governs only the steady-state behavior, $\mathcal{A}(c)$ is responsible not only for the particle velocity but also for the entire time evolution; for example, in the exponent or the modified Bessel function term in Eq. (33), there

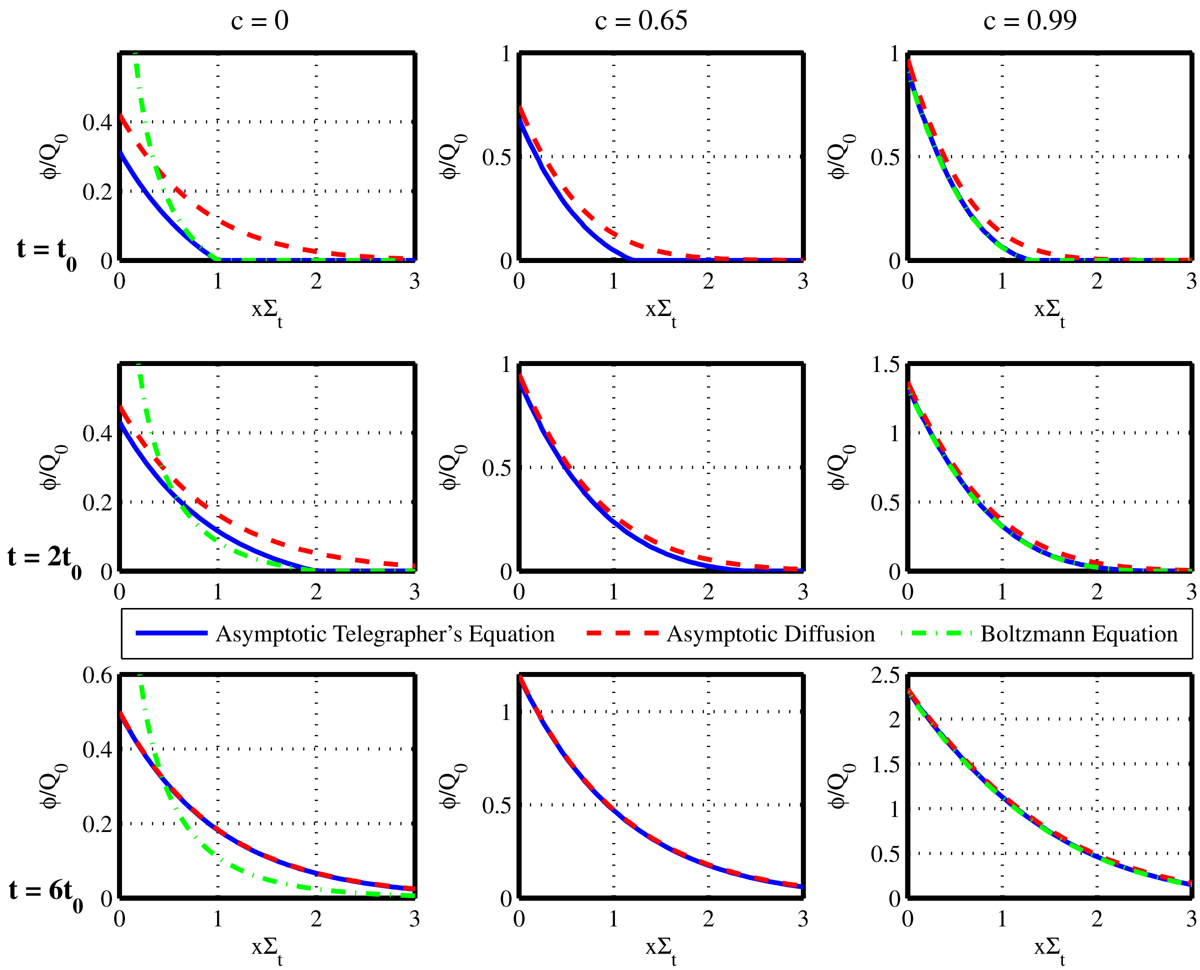

Fig. 3. The scalar flux in a source problem in a purely absorbing medium ($c = 0$) resulting from the asymptotic Telegrapher's equation (solid lines) using Eq. (33) and in the asymptotic diffusion approximation (dashed lines) at different times, with $c = 0.65$ using Eq. (33), and in a highly scattering medium ($c \approx 1$); $t_0 \equiv (v\Sigma_t)^{-1}$, and $t = 6t_0$ represents very long times. The exact Boltzmann solution for $c = 0$ is taken from Eq. (56), and for $c \approx 1$ it is taken as the first term of Eq. (58).

$\mathcal{A}(c) = 1$ would not necessarily be the preferable choice, as we have shown in Eq. (58). Therefore, a careful examination should be made over a variety of albedo-dependent problems with the exact solutions besides the analytic solution of Eq. (33) (using S_N or Monte Carlo calculations) to determine the optimal $\mathcal{A}(c)$.

IV.B. Comparison with Semianalytic Solutions

In addition to the scalar flux in the time-independent source, which tends to the steady-state problem, one can test the asymptotic Telegrapher's equation approximation with an exact semianalytical solution, e.g., the AZURV1 benchmark.[42] The AZURV1 semianalytic solution makes use of the full Green function (in both space and time) in one-dimensional sources in infinite homogeneous media.

The source now is the full Green function (both in space and time), i.e., $Q(x,t) = Q_0\delta(x)\delta(t)$ (instead of the time-independent source that was considered earlier, $Q(x) = \delta(x)$, which tends to the steady-state solution). This source compels us to modify the Telegrapher's equation to include time-dependent sources, which we ignored earlier. One can easily see from a simple derivation of Eqs. (31) that Eq. (32) becomes

$$\left[\frac{\mathcal{A}}{v}\right]\frac{\partial^2\phi(x,t)}{\partial t^2} - [v]\frac{\partial^2\phi(x,t)}{\partial x^2} + [\mathcal{A}\Sigma_a + \mathcal{B}\Sigma_t]\frac{\partial\phi(x,t)}{\partial t} + [v\mathcal{B}\Sigma_a\Sigma_t]\phi(x,t) = v\mathcal{B}\Sigma_t Q(x,t) + \mathcal{A}\frac{dQ(x,t)}{dt} \tag{59}$$

with a dQ/dt term (see also in Refs. 29, 30, 43, and 44).

We solve the Telegrapher's equation (both asymptotic and classic) by substituting $\phi(x,t) = \varphi(x,t)e^{-v\Sigma_t t}$ and applying the Fourier transform in space and the Laplace transform in time. The Telegrapher's equation solution is[31] (see footnote b on p. 20)

$$\phi(x,t) = \frac{Q_0\sqrt{\mathcal{A}}}{4}\, e^{-(v/2)(\Sigma_a + (\mathcal{B}/\mathcal{A})\Sigma_t)t} \left\{ \delta\left(t - \frac{\sqrt{\mathcal{A}}}{v}x\right) + \delta\left(t + \frac{\sqrt{\mathcal{A}}}{v}x\right) \right.$$
$$- \left(\Sigma_a - \frac{\mathcal{B}}{\mathcal{A}}\Sigma_t\right) v \cdot I_0\left[\frac{v}{2}\left(\Sigma_a - \frac{\mathcal{B}}{\mathcal{A}}\Sigma_t\right)\sqrt{t^2 - \mathcal{A}\frac{|x|^2}{v^2}}\right] \cdot H\left(t - \sqrt{\mathcal{A}}\frac{|x|}{v}\right)$$
$$\left. + \left(\Sigma_a - \frac{\mathcal{B}}{\mathcal{A}}\Sigma_t\right) vt \cdot \frac{I_1\left[\frac{v}{2}\left(\Sigma_a - \frac{\mathcal{B}}{\mathcal{A}}\Sigma_t\right)\sqrt{t^2 - \mathcal{A}\frac{|x|^2}{v^2}}\right]}{\sqrt{t^2 - \mathcal{A}\frac{|x|^2}{v^2}}} \cdot H\left(t - \sqrt{\mathcal{A}}\frac{|x|}{v}\right) \right\} , \tag{60}$$

where

$I_n(x')$ = modified Bessel function of the first kind of order n

$H(x,t)$ = Heaviside step function

$\delta(x,t)$ = Dirac delta function.

For the classic Telegrapher's equation, $\mathcal{A} = \mathcal{B} = 3$, and for the asymptotic Telegrapher's equation, $\mathcal{A}(c)$ and $\mathcal{B}(c) = 1/D_0(c)$ are determined from Eqs. (54) and (25), respectively. This solution agrees with previous work[43,44] (with $\mathcal{A} = \mathcal{B} = 3$, of course).

The corresponding asymptotic diffusion solution for the case of $Q(x,t) = Q_0\delta(x)\delta(t)$ is found in a similar way [but with $\phi(x,t) = \varphi(x,t)e^{-v\mathbf{\Sigma_a}t}$ instead][31]:

$$\phi(x,t) = \frac{Q_0}{2}\sqrt{\frac{\mathcal{B}v\Sigma_t}{\pi t}}\, e^{-(\mathcal{B}\Sigma_t|x|^2/4vt)} \cdot e^{-\Sigma_a vt} \ , \tag{61}$$

where $\mathcal{B}(c) = 1/D_0(c)$.

We focus here on the purely scattering case $c = 1$ since the scalar flux for a general c can be written as a function of the scalar flux resolving the case for $c = 1$ (Refs. 18 and 45) (at least in the problem of an infinite homogeneous source-free medium):

$$\phi^{(c)}(x,t) = ce^{-(1-c)v\Sigma_t t}\phi^{(1)}(cx,ct) \ ; \tag{62}$$

i.e., checking our solution for $c = 1$ is valid for all c (for this problem).

In Fig. 4 we compare our asymptotic Telegrapher's equation approximation for $c = 1$ at different times [Eq. (60) with $\mathcal{A} = \frac{3}{5}$ and $\mathcal{B} = 3$] with the exact transport semianalytic solutions, obtained from different sources[42,44–47] (the AZURV1 benchmark). In addition, we also plot the classic Telegrapher's equation solution [Eq. (60) with $\mathcal{A} = \mathcal{B} = 3$] and the time-dependent asymptotic diffusion approximation [Eq. (61), which for $c = 1$ tends to the classical diffusion approximation], which are the two natural competitor approximations to our approximation.

Since the exact solution is performed in discrete values,[42,44–47] we used Boffi's approximation,[45,48] which is based on a series expansion around the diffusion solution solving the pole portion of the transport solution. We note that we should use this approximation only as a rough guide since this approximation is not exact, especially for small x at small t, yielding a negative (nonphysical) scalar flux (Fig. 4, which is caused from the diffusive behavior of this approximation).

We can see from Fig. 4 that at early and intermediate times, the classic Telegrapher's equation approximation is inferior to both the asymptotic diffusion approximation and the asymptotic Telegrapher's equation. Even at $t = 7t_0$ the wrong particle velocity affects the solution (manifesting as a "knee"-shaped curve). This is due to the well-known proposition that the P_1 approximation is less accurate than the time-dependent diffusion equation, despite its hyperbolic nature (for example, see Ref. 41). In the limit of $t \to \infty$, all the approximations are in agreement with the exact solution.

At early times, the asymptotic Telegrapher's equation yields a scalar flux similar to the asymptotic diffusion approximation, except the tail of the distribution function at infinite distance that exists in diffusion solutions and does not exist in hyperbolic solutions. The advantages of the asymptotic Telegrapher's equation are both the quite good accuracy of the asymptotic diffusion approximation in asymptotic regimes (i.e., far from the

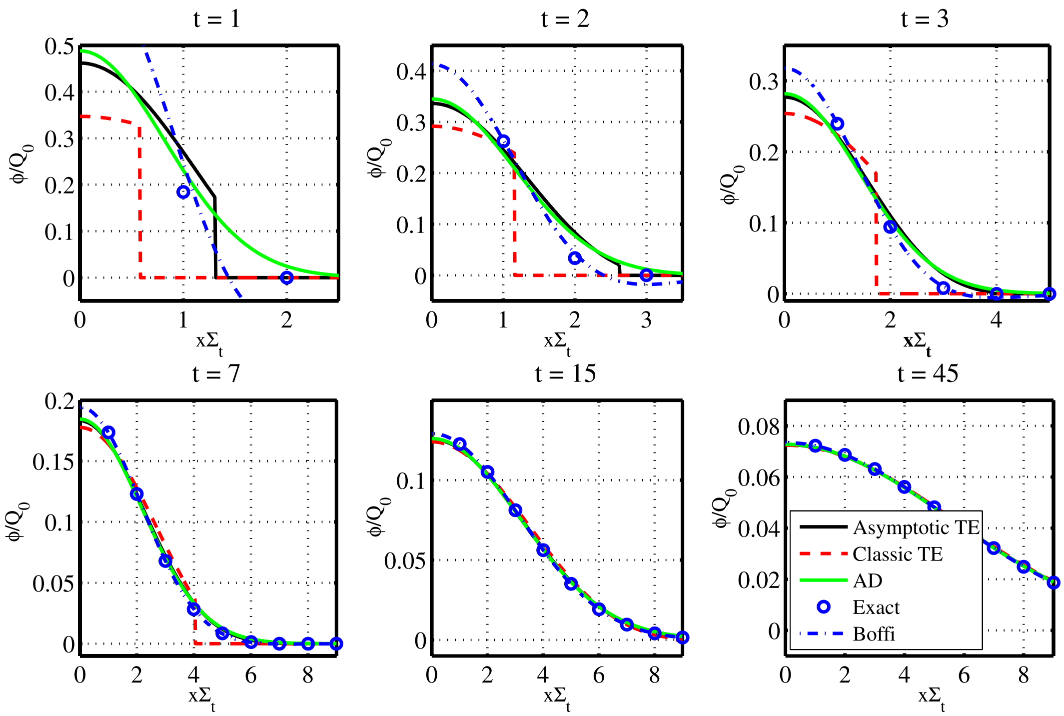

Fig. 4. The scalar flux in the AZURV1 benchmark in a purely scattering medium ($c = 1$) resulting from the asymptotic Telegrapher's equation (solid lines) and the classic Telegrapher's equation (gray solid lines) using Eq. (60) and in the asymptotic diffusion approximation (dashed lines) at different times. The exact solution for the Boltzmann equation is taken from Refs. 42 and 44 through 47, while we used Boffi's approximation[48] to receive a solid line along the discrete values of the exact solution (dash-dotted lines).

knee around $x \approx vt$ and the good accuracy near the knee (around $x \approx vt$). (In the exact solution there is no knee because the scalar flux must be continuous. The asymptotic part itself may be discontinuous; thus, the asymptotic Telegrapher's equation yields a knee-shaped solution.) For a complementary picture, we plot the scalar flux in the t plane in Fig. 5.

As before, we can see that at long times, all the approximations are in good agreement with the exact solution (Fig. 5a). At early times (Fig. 5b), the advantage of the asymptotic Telegrapher's equation over the asymptotic diffusion solution is obvious. We also checked our solution for different values of c, and the picture (both Figs. 4 and 5) remains quite the same, as expected [Eq. (62)]. In fact, the location of the knee is better as c decreases [because $\mathcal{A}(c) \rightarrow 1$]. In this section we checked the plane source problem, but the conclusion is expected to be similar in any one-dimensional geometry.

V. DISCUSSION

An asymptotic Telegrapher's equation approximation is derived that describes both the asymptotic steady-state behavior (for sufficient large t) and the time evolution behavior. The derivation was based on the asymptotic behavior of the time-dependent Boltzmann equation, in time as well as in space. This approximation should allow us to describe the (asymptotic) behavior of the full Boltzmann equation in both space and time, at the cost of solving the Telegrapher's equation only (or P_1 equations), just as the asymptotic diffusion approximation has done in the steady-state case (at the cost of solving a diffusion equation). Solving the Telegrapher's equation or the P_1 equations is of course much simpler than solving the full Boltzmann equation, analytically or numerically.[9,19–24,30,32,33,43]

In Sec. IV the asymptotic Telegrapher's equation was compared to the classic Telegrapher's equation and the time-dependent asymptotic diffusion equation. Our approximation yields a better fit for the scalar flux for both the time-independent source term, which was checked analytically in extreme cases, and to semianalytical published solutions[42,44–47] for the full Green function. It seems that the advantages of the asymptotic Telegrapher's equation are both quite good accuracy of the asymptotic diffusion approximation in asymptotic regimes and good accuracy near the knee (around $x \approx vt$).

This approach, i.e., finding modified coefficients for the Telegrapher's equation by using the asymptotic behavior of the time-dependent Boltzmann equation, was

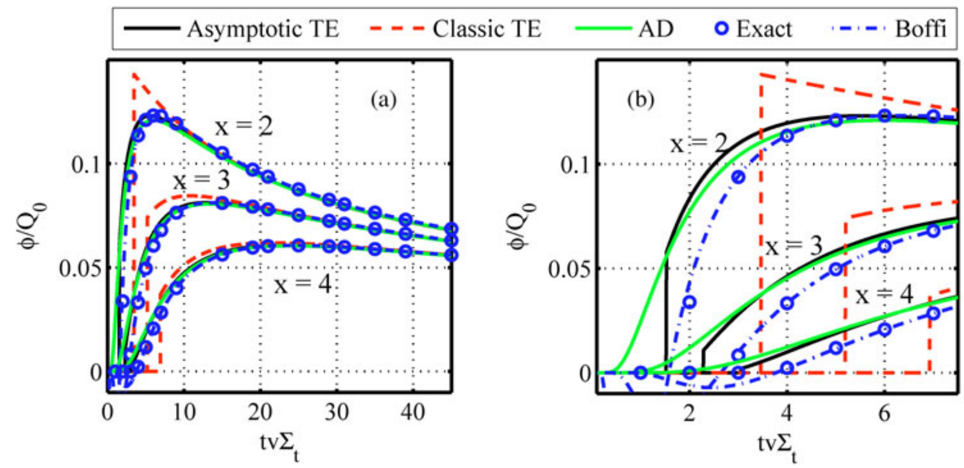

Fig. 5. (a) The scalar flux in the AZURV1 benchmark in a purely scattering medium ($c = 1$) resulting from the asymptotic Telegrapher's equation (solid lines) and the classic Telegrapher's equation (gray solid lines) using Eq. (60) and in the asymptotic diffusion approximation (dashed lines) for different x. The exact Boltzmann solution is taken from Refs. 42 and 44 through 47, while we used Boffi's approximation[48] to receive a solid line along the discrete values of the exact solution (dash-dotted lines). (b) Zoom on early times.

also used in work by Hoenders and Graaff[25] but with a different rationale and different derivation (they used a certain ansatz for the scalar flux, both in space and time). They have shown an equivalent value to $\sqrt{\mathcal{A}(c)}$ (in our terminology) that is very similar to the solid curve in Fig. 2b [by using similar approximation for $D_0(c)$ introduced in Eq. (23), which is accurate near $c \approx 1$ and thus, received a similar formula to Bengston's work[10,41]]. The similarity between the results of these two works, despite the different derivation, reinforces the accepted results.

Nevertheless, there are some advantages in our derivation over the derivation introduced in Ref. 25. First, the derivation that was introduced here for the asymptotic Telegrapher's equation approximation, or the asymptotic P_1 approximation, clarifies the regime of the allowed approximation; Eq. (31a) is exact (the conservation law) and, thus, should remain untouched, while Eq. (31b) is an approximate equation; thus, this is the equation on which we must focus should we want a more accurate approximation. Second, this analysis shows us that using the resulting $\mathcal{A}$ and B is the best approximation to the Boltzmann equation that P_1 equations can give. A more accurate approximation will have to include higher derivatives of the scalar flux by time, corresponding to terms proportional to $\mathcal{O}(s^2)$, $\mathcal{O}(s^3)$, and so on, which are beyond the P_1 regime. In addition, the rationale introduced here clarifies that yielding the correct particle velocity does not summarize the entire time evolution behavior, as mentioned before.

Moreover, the generalization to multidimensional general geometry in our derivation is as simple as the generalization for the steady-state problem, which is well known. This generalization appears, for example, in Chapter IV of Ref. 8 or in Chapter III(3) of Ref. 4, and we shall not repeat it here. On the other hand, the derivation in Ref. 25 is introduced only for one-dimensional slab geometry.

Although the derivation is valid for a multidimensional general geometry, it was examined here for a simple source problem in a one-dimensional slab geometry. Additional work should be done in two- and three-dimensional geometries to examine the validity of our asymptotic Telegrapher's equation (P_1) approximation by comparison to the full Boltzmann equation solutions and flux-limiter solutions in multidimensional geometries.

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