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Shay I. Heizler

Department of Physics, Bar-Ilan University, Ramat-Gan, Israel
Department of Physics, Nuclear Research Center-Negev, Beer Sheva, Israel

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The Asymptotic Telegrapher’s Equation (P₁) Approximation for Time-Dependent, Thermal Radiative Transfer

Shay I. Heizler

Department of Physics, Bar-Ilan University, Ramat-Gan, Israel; Department of Physics, Nuclear Research Center-Negev, Beer Sheva, Israel

We develop the asymptotic P₁ approximation for the time-dependent thermal radiative transfer equation for a multidimensional general geometry. Careful derivation of the asymptotic P₁ equations, directly from the time-dependent Boltzmann equation, yields a particle velocity that is closer (v \approx 0.91c) to the exact value of c but is based on an asymptotic analysis rather than diffusion theory (infinite velocity) or conventional P₁ theory (which gives rise to the Telegrapher’s equation, v = 1/√3c \approx 0.577c). While this approach does not match the exact value of c as does the P₁/3 method, the latter method is an ad hoc approach that has not been justified on theoretical grounds. This article provides the theoretical justification for the almost-correct value of c that yields improved results for the well-known (one-dimensional) Su-Olson benchmark for radiative transfer, for which we obtain a semi-analytic solution in the case of local thermodynamic equilibrium. We found that the asymptotic P₁ approximation yields a better solution than the diffusion, the classic P₁, and the P₁/3 approximations, yielding the correct steady-state behavior for the energy density and the (almost) correct particle velocity.

Keywords Boltzmann equation; kinetic theory; diffusion equation; radiative transfer; transport theory

I. INTRODUCTION

The diffusion (or the Eddington) approximation is a well-known approximation for the Boltzmann (transport) equation (Pomraning, 1973) and is extensively used, especially in radiative transfer (RTE). However, the diffusion approximation yields the wrong time-description because of its parabolic nature; the diffusion approximation yields an infinite particle velocity. The classic P₁
approximation, which gives rise to the hyperbolic Telegrapher’s equation, replaces one problem with another; instead of infinite velocities, we get a wrong finite particle velocity, $c/\sqrt{3}$.

To tackle this problem, a large variety of approximation methods were offered, such as Flux-Limiters solutions (Winslow, 1968; Levermore, 1979; Levermore and Pomraning, 1981; Pomraning, 1981, 1984; Su, 2001) or Variable Eddington-Factors approximations (Pomraning, 1981; Su, 2001) (in $P_1$ form), which converted the diffusion equation into a nonlinear partial differential equation. Despite the great success and the wide use of these approximations (Olson et al., 2000), the nonlinearity of the equation makes finding the solution much harder, and analytic solutions are seldom available. Moreover, these solutions define a gradient-dependent diffusion coefficient (or Eddington factors), which is trivial to apply in one-dimensional geometries. But, in multidimensional problems (and especially in curvilinear geometries in general meshes), it requires using multiple diffusion coefficients for each cell (for each boundary of the cell), distorting the radiation fields (usually because of the shape of the mesh). This problem demonstrates the advantage of using linear approximations like diffusion or $P_1$.

A certain linear, ad hoc approximation, was offered, called the $P_{1/3}$ approximation, to yield the exact particle velocity (Olson et al., 2000; Morel, 2000; Simmons and Mihalas, 2000), by a superposition of diffusion and $P_1$ (with some weights). It is also accurate to first order in a first-order accuracy compared to the exact equation in the diffusion-limit asymptotics (Morel, 2000). This approximation was verified against some problems (Olson et al., 2000), especially the famous Su-Olson benchmark (Olson et al., 2000; Su and Olson, 1996, 1997, 1999) for a propagating Marshak wave (Marshak, 1958). One of the conclusions of (Olson et al., 2000) was indeed that the “$P_{1/3}$ is recommended over any of the flux-limited diffusion theories as a general-purpose choice,” in the gist of the previous paragraph ending. Unfortunately it is nothing more than an ad hoc approximation, and is not based on a physical derivation. (It should be noted that in one-dimensional problems, a certain $P_1$ approximation can be derived that predicts the exact transport behavior, based on the forward-backward approximation (Porra et al., 1997), which is similar to the $S_2$ approximation with $\mu = \pm 1$. Unfortunately, the method cannot be generalized to multiple dimensions.)

Recently, a new approach was offered based on an asymptotic development (both on space and time) of the time-dependent Boltzmann equation, called the asymptotic $P_1$ approximation (or the asymptotic Telegrapher’s equation approximation) (Heizler, 2010). This approximation yields the correct steady-state eigenvalue of the asymptotic diffusion approximation (which was developed by Frankel and Nelson, 1953, and expanded by Case et al., 1953, 1967 and by Davison and Sykes, 1958), and the (almost) correct time evolution, such as the particle velocity (the wave-front evolution). In addition, a recent work has
shown that the asymptotic $P_1$ approximation shares the same asymptotics in the diffusion limit with the $P_2$ approximation in one-dimensional slab geometry, or the simplified $P_2$ ($SP_2$) approximation in a general geometry (Heizler and Ravetto, 2012). This approximation was developed mainly to deal with neutron transport, and it is worthwhile to develop and test it in radiative transfer problems, where the use of diffusion-models is vast. Similar studies in optics (Hoenders and Graaff, 2005, 2008) and in charged-particle transport (Gombosi et al., 1998; Kaghashvili et al., 2004), have reached similar (but partial) conclusions. As we see it, one of the most important goals of this article is to try to give a rigorous physical basis for the ad hoc $P_{1/3}$ approximation, based on asymptotic derivation of the exact time-dependent transport equation (beside the empirical success (Olson et al., 2000) and the asymptotic accuracy check in the diffusion-limit (Morel, 2000)).

One of the most effective ways to test a new approximation is by comparing it to known analytical benchmarks (Olson et al., 2000; Su and Olson, 1996; 1997; 1999; Pomraning, 1979; Ganapol, 1979; Shokair and Pomraning, 1981; Ganapol and Pomraning, 1983). We focus (for illustrative purposes) on a LTE-adjustment (local thermodynamic equilibrium) to the Su-Olson (1997) benchmark. We are especially interested in the recently published work that presents an analytic solution for the nonequilibrium Su-Olson benchmark using the $P_1$ approximation (McClarren et al., 2008). It is important to note the this article does not focus on comparing the accuracy of several approximations for the transport equation. For this kind of comparisons, see, for example (Pomraning (1981); Su (2001); and Olson et al. (2000). This article focuses on presenting the asymptotic $P_1$ approximation for radiative transfer and presents a basic examination.

The examination of the asymptotic $P_1$ approximation will follow the methodology in McClarren et al., (2008), but with an opposite rationale. In McClarren et al. analytic solutions of the classic $P_1$ approximation were found for benchmarking a general spherical harmonics ($P_N$) code (an exact transport solution in the limit $N \to \infty$). Here, we intend to show via a semi-analytical solution for a LTE-adjustment of the Su-Olson benchmark that the asymptotic $P_1$ approximation reproduces the main features of a full transport solution at the cost of solving $P_1$ equations.

It is important to note that the derivation here for the asymptotic $P_1$ approximation to the RTE is a full multidimensional (grey) general geometry derivation. Moreover, the generalization to multi-energy groups treatment is trivial, using an energy-dependent albedo (Pomraning, 1973). The one-dimensional Su-Olson benchmark is used here only for illustrative purposes, for testing the new approximation, but the major conclusions concerning the particle velocity and the total time-evolution behavior are general.

The work is structured as follows: First, in Section II we present the $P_1$, diffusion and asymptotic $P_1$ equations for radiative transfer, using (McClarren
et al., 2008) notations. Next, in Section III we present a rigorous derivation of the asymptotic $P_1$ equations in multidimensional general geometry (for an infinite, homogeneous medium) from an asymptotic treatment of the monoenergetic time-dependent Boltzmann equation. It also presents an asymptotics accuracy check of the asymptotic $P_1$ approximation in the diffusion limit. In Section IV we develop the Green function for the asymptotic $P_1$ equations for the LTE case. Then, in Section V we present the capabilities of the new approximation, by comparing the one-dimensional slab geometry Su-Olson benchmark with a LTE-solution of the asymptotic $P_1$ equations for the Su-Olson benchmark. Section VI presents a short discussion. In general, we mark explicitly when and where an LTE assumption is made.

II. $P_1$, DIFFUSION, AND ASYMPTOTIC $P_1$ EQUATIONS FOR RADIATIVE TRANSFER

In this section we introduce the basic notations of radiative transport. This part is crucial for understanding the basic rationale of the new approximation. In general, we follow the notations of (McClarren et al., 2008). The radiation transport equation for multidimensional general geometry for grey ("monoenergetic") radiation is:

$$\frac{1}{c} \frac{\partial I}{\partial t} + \hat{\Omega} \cdot \vec{\nabla} I = \sigma (B - I) + S,$$

where $I$ is the specific intensity that is a function of space, angle ($\hat{\Omega}$, a unit vector represents the particle direction of motion) and time, $\sigma$, is the absorption opacity (for simplicity, we neglect here the scattering opacity), which in general depends on space and the specific intensity (thus, the problem in general is nonlinear), $S$ is a prescribed, isotropic, and inhomogeneous source and $c$ is the speed of light. $B$ is the local black-body radiation emitted from the material with a given temperature $T_m$ (we assume that the electrons and the ions of the material are in equilibrium, and there is a defined "temperature," $T_m$, for the material), which is function of space and time:

$$B = \frac{ac}{4\pi} T_m^4,$$

where $a$ is the black-body constant defined by:

$$a = \frac{4\sigma_{sb}}{c},$$

and $\sigma_{sb}$ is the Stefan-Boltzmann constant.
Applying angular integration to Equation (1), we get the zero moment of the transport equation, the conservation law:

\[
\frac{1}{c} \frac{\partial E}{\partial t} + \frac{1}{c} \bar{\nabla} \cdot \bar{F} = \sigma (aT_m^4 - E) + S.
\] (4)

where \( E \) is the energy density and is defined as the zero’s moment of the specific intensity \( I \) (by a factor of \( 1/c \)):

\[
E = \frac{1}{c} \int_{4\pi} I(\hat{\Omega})d\hat{\Omega}.
\] (5)

and \( \bar{F} \) is the radiation flux, the first moment of the specific intensity:

\[
\bar{F} = \int_{4\pi} I(\hat{\Omega})\hat{\Omega}d\hat{\Omega}.
\] (6)

Equation (4) is an exact equation.

Along with the equations for the radiation energy, the complementary equation for the material is:

\[
\frac{C_{v,m}(T_m)}{c} \frac{\partial T_m}{\partial t} = \sigma (E - aT_m^4),
\] (7)

where \( C_{v,m}(T_m) \) is the heat capacity of the material. In this study we assume that the heat capacity has this functional shape:

\[
C_{v,m} = aT_m^3.
\] (8)

This is a common choice for the heat capacity for analytic solutions (Su and Olson, 1996, 1997, 1999; Pomraning, 1979; McClarren et al., 2008). It is convenient to define a material energy \( U_m \) as a function of the material temperature \( T_m \):

\[
U_m = aT_m^4 = \frac{4\pi}{c} B.
\] (9)

With Equation (9) and with \( \varepsilon \equiv 4a/\alpha \), Equations (7) and (8) take this form:

\[
\frac{1}{c\varepsilon} \frac{\partial U_m}{\partial t} = \sigma (E - U_m)
\] (10)

**A. Grey \( P_1 \) Approximation**

Assuming that the specific intensity is decomposed from its two first moments (by the Legendre series) and operating \( \int_{4\pi} \hat{\Omega}d\hat{\Omega} \) over Equation (1) yields (Pomraning, 1973):

\[
\frac{1}{c} \frac{\partial \bar{F}}{\partial t} + \frac{c}{3} \bar{\nabla} E + \sigma \bar{F} = 0.
\] (11)
Equation (11) is an approximate equation for the first moment of the transport equation. Together with the conservation law (Equation (4) and Equation (10)), these equations comprise a closed set of equations for both the radiation and the material energy, the classic grey $P_1$ approximation (Pomraning, 1973; McClarren et al., 2008).

B. Grey Diffusion Approximation

To introduce the diffusion approximation, one must assume also that the derivative of $F$ with respect to time is negligible, i.e., $(1/|F|)(\partial |F|/\partial t) \ll c\sigma$. With this additional assumption, Equation (11) takes a form of a Fick’s law:

$$\vec{F} = -\frac{cD_0}{\sigma} \vec{\nabla} E,$$

with $D_0 \equiv 1/3$ as the dimensionless diffusion coefficient. Substituting Fick’s law (Equation (12)) into the conservation law (Equation (4)) yields a diffusion equation:

$$\frac{1}{c} \frac{\partial E}{\partial t} - \vec{\nabla} \cdot \left( \frac{cD_0}{\sigma} \vec{\nabla} E \right) = \sigma (aT_m^4 - E) + S.$$  

Equation (13) along with the material energy equation (Equation (10)) comprise the grey diffusion approximation. The grey diffusion approximation (Equations (13) and (10)) is usually expressed in terms of the temperature (Zel’dovich and Raizer, 2002), and is called the “two-temperature” diffusion approximation. Using the radiation temperature, defined as $T_R \equiv \sqrt[4]{E/a}$, and the notation for the thermal-conductivity parameter:

$$\chi(T) \equiv \frac{4caT_R^3D_0}{\sigma} = \frac{16\sigma sb T_R^3}{3\sigma},$$

Equation (13) can be written as:

$$\frac{C_v R(T)}{c} \frac{\partial T_R}{\partial t} - \vec{\nabla} \cdot (\chi(T_R)\vec{\nabla} T_R) = \sigma a (T_m^4 - T_R^4) + S$$

with $C_v R(T) \equiv 4aT_R^2$ as the “radiation heat capacity.”

C. LTE Diffusion Approximation

In the case of local thermodynamic equilibrium (LTE), the radiation is in local equilibrium with the material, $E \approx U_m$. In this case, by adding Equations (13) and (10), we get a single equation, the LTE diffusion equation:

$$\frac{1}{c} \left( \frac{1 + \varepsilon}{\varepsilon} \right) \frac{\partial E}{\partial t} - \vec{\nabla} \cdot \left( \frac{cD_0}{\sigma} \vec{\nabla} E \right) = S.$$  

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In terms of temperature, the LTE diffusion approximation (where the radiation temperature is equal to the material temperature) is defined by adding Equation (15) with Equation (7) (assuming $E \approx U_m$):

$$\frac{C_v'(T)}{c} \frac{\partial T}{\partial t} - \nabla \cdot (\kappa(T) \nabla T) = S,$$

where $C_v'(T)$ is the total heat capacity and it is composed of both the material heat capacity and the radiation heat capacity:

$$C_v'(T) \equiv C_{v,m}(T) + C_{v,R}(T) = \alpha T^3 + 4aT^3 = \alpha T^3(1 + \varepsilon).$$

D. Asymptotic $P_1$ Approximation

As mentioned before, the classic $P_1$ approximation is composed of two equations that yield the wrong particle velocity, $c/\sqrt{3}$. The first one is exact, the conservation law (Equation (4)), and the second is approximated (Equation (11)) and contains the factor of 3. Thus, the rationale of the asymptotic $P_1$ approximation (Heizler, 2010) motivate us to find a modified equation of this form (for an elaborate discussion, see (Heizler, 2010)):

$$\frac{A}{c} \frac{\partial \tilde{F}}{\partial t} + c \tilde{\nabla} E + \sigma B \tilde{F} = 0,$$

where $A$ and $B$ are two media-dependent parameters that are determined by an asymptotic derivation of the exact time-dependent Boltzmann equation (see Section III). $B$ determines the steady-state behavior and equals to $B = 1/D_0$ according to the asymptotic diffusion approximation. In the LTE case $B = 3$, but it can be modified to include general media features. $A$ determines the time evolution, including particle velocity. Of course, $A = B = 3$ reproduces the classic $P_1$ approximation.

Substituting the modified $P_1$ equation (Equation (19)) into the conservation law (Equation (4)) yields the asymptotic Telegrapher’s equation (Heizler, 2010) for radiation transfer:

$$\frac{A}{c} \frac{\partial^2 E}{\partial t^2} - c \nabla^2 E + \sigma \frac{\partial E}{\partial t}(B + A) + \sigma^2 c B E = \left( \sigma c B + A \frac{\partial}{\partial t} \right)(\sigma U_m + S).$$

Applying the Laplace transform to Equation (19) yields an equation with the form of Fick’s law:

$$\tilde{F} = -cD(s) \cdot \tilde{\nabla} E$$

with this $s$-dependent diffusion coefficient:

$$D(s) \equiv \frac{c}{As + Bc\sigma}.$$
Equation (22) is helpful in finding the asymptotic constants $A$ and $B$ (see Section III).

**E. $P_{1/3}$ Approximation**

Setting (ad hoc) $B = 3$ and $A = 1$ (called the $P_{1/3}$ approximation, since dividing Equation (19) with $B$, yields a $1/3$ factor which multiplies the derivative of $F$ with respect to $t$), yields the correct particle velocity (Olson et al., 2000) (This also matches the rationale of (Heizler, 2010), that the factor of 3 in the time-derivative in the classic $P_1$ approximation yields a particle velocity that is off by a factor of $1/\sqrt{3}$). In other words, the $P_{1/3}$ approximation can be defined by adding the $P_1$ equation with a weight of one third to the diffusion equation with a weight of two thirds (again, ad hoc). In this article we intend to give the physical basis for setting a different value for $A$, other than 3 (the classic $P_1$ approximation), based on an asymptotic derivation of the time-dependent Boltzmann equation, which will be as close to $A = 1$ (as far as the particle velocity is concerned).

**III. Derivation of the Asymptotic $P_1$ Approximation from the Time-Dependent RTE**

In this section we derive by asymptotic derivation a modified $P_1$ equation, finding the constants $A$ and $B$ for a general non-LTE case. The LTE case will be discussed in the end as a separate case; there, a fully analytical derivation can be obtained. This analysis is similar to the analysis that is presented in Heizler (2010) for neutronics for the one-dimensional slab geometry case. The derivation here is presented for a multidimensional general geometry case (by general, we mean that we do not impose any specific symmetry on the system), based on the time-independent derivation, presented in Pomraning (1973; chapter III(3)). The asymptotic analysis, valid for an infinite, homogeneous medium, is used to provide the coefficients for general problems.

We start with the time-dependent monoenergetic source-free Boltzmann equation for radiation:

$$\frac{1}{c} \frac{\partial I}{\partial t} + \hat{\Omega} \cdot \vec{v} I = \sigma \left( \frac{c}{4\pi} U_m - I \right).$$

(23)

We use the common definition of the effective albedo, $\tilde{\omega}_{\text{eff}}$ (or $\gamma(z)$ in Equations (3.97–101) in Pomraning (1973), without scattering), i.e., the ratio between the material energy density and the radiation energy density (see also the definition of $\beta$ in (Fleck and Cumming, 1971)):

$$\tilde{\omega}_{\text{eff}} \equiv \frac{U_m}{E} = \frac{4\pi B}{cE}.$$  

(24)

1953), the albedo is referred to as a parameter (even in its multigroup definition), since the derivations are asymptotic on space, and in our case, both on space and time. Using Equation (24), Equation (23) takes this form:

\[
\frac{1}{c} \frac{\partial I}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I + \sigma I = \sigma \frac{c \tilde{\omega}_{\text{eff}}}{4\pi} E. \tag{25}
\]

Applying the Laplace transform to Equation (25) yields:

\[
\frac{s}{c} I' + \vec{\Omega} \cdot \vec{\nabla} I' + \sigma I' = \sigma \frac{c \tilde{\omega}_{\text{eff}}}{4\pi} E'(s). \tag{26}
\]

Defining a modified \(s\)-dependent cross-section:

\[
\sigma'(s) = \sigma + \frac{s}{c}. \tag{27}
\]

Equation (26) can be re-arranged as a modified time-independent radiative Boltzmann equation:

\[
\vec{\Omega} \cdot \vec{\nabla} I'(s) + \sigma'(s) I'(s) = \sigma \frac{c \tilde{\omega}_{\text{eff}}}{4\pi} E'(s) \tag{28}
\]

where \(c'(s)\) is a modified \(s\)-dependent effective albedo:

\[
c'(s) = \frac{\sigma \tilde{\omega}_{\text{eff}}}{\sigma'(s)} = \frac{\tilde{\omega}_{\text{eff}}}{1 + \frac{s}{\sigma}}. \tag{29}
\]

Since Equation (28) has a form of the time-independent Boltzmann equation, the derivation is straightforward and we will go over it briefly. Using the definition of the energy density \(E\) as the zero moment of the specific intensity \(I\) (Equation (5)), we use the \(s\)-dependent Pierls's integral equation as an corresponding to Equation (28) (Pomraning, 1973):

\[
E(r,s) = c'(s)\sigma'(s) \int_V d\vec{r}' e^{-\sigma'(s)|\vec{r} - \vec{r}'|} E(\vec{r}', s). \tag{30}
\]

Since we are interested only in the asymptotic solutions, we take the volume \(V\) to be infinite, and we disregard the boundary conditions. We assume a solution for Equation (30) of this kind:

\[
E'(s, \vec{r}) = \frac{1}{c} e^{\sigma'(s)\tilde{\omega}(s) \vec{r}}, \tag{31}
\]

where \(\tilde{\omega}(s) = \omega(\vec{u})\) is an \(s\)-dependent steady-state eigenvalue with \(\vec{u}\) as a unit vector. Substituting Equation (31) in Equation (30) and carrying out the integration over the space yields:

\[
e^{\sigma'(s)\tilde{\omega}(s) \vec{r}} = \frac{c'(s)}{2\tilde{\omega}_0(s)} \ln \left( \frac{1 + \tilde{\omega}'(s)}{1 - \tilde{\omega}'(s)} \right) e^{\sigma'(s)\tilde{\omega}(s) \vec{r}}. \tag{32}
\]
Reducing the exponents yields the well-known closed equation for the \(s\)-dependent eigenvalues:

\[
\frac{2}{c'(s)} = \frac{1}{\nu_0'(s)} \ln \left( \frac{1 + \nu_0(s)}{1 - \nu_0(s)} \right) .
\] (33)

Equation (33) has only two solutions with \(|\nu'(s)| < 1\), \(\pm \nu_0'(s)\), and they are called the asymptotic solutions. There is a range of solution for Equation (33) with \(|\nu'(s)| > 1\). These solutions are called transient solutions since they decay relatively fast, compared to the asymptotic solutions \(\pm \nu_0'(s)\). The solution of Equation (33) is calculated numerically and tabulated in Case et al. (1953).

A general asymptotic solution will be a superposition of solutions, of this kind:

\[
E'(s, \bar{r}) = \frac{1}{c} \int d\bar{u} f(\bar{u}) e^{\sigma'(s)\nu'(s)\bar{u} \cdot \bar{r}} ,
\] (34)

where \(f(\bar{u})\) is an arbitrary function. The asymptotic \(s\)-dependent specific intensity can be calculated via the integral transport equation (Pomraning, 1973):

\[
I'(s, \bar{r}, \bar{\Omega}) = \frac{c'(s)\sigma'(s)c}{4\pi} \int_0^\infty dk e^{-\sigma'(s)k} E'(s, \bar{r} - k\bar{\Omega}) .
\] (35)

Substituting Equation (34) in Equation (35) yields:

\[
I'(s, \bar{r}, \bar{\Omega}) = \frac{c'(s)\sigma'(s)c}{4\pi} \int d\bar{u} f(\bar{u}) e^{\sigma'(s)\nu'(s)\bar{u} \cdot \bar{r}} \int_0^\infty dk e^{-\sigma'(s)(1 + \nu'(s)\bar{u} \cdot \bar{\Omega})k}.
\] (36)

The \(i\)th component of \(F\) (the radiation flux, as the first moment of the specific intensity) is:

\[
F'^{(i)}(s, \bar{r}) = \frac{c'(s)}{4\pi} \int d\bar{u} f(\bar{u}) e^{\sigma'(s)\nu'(s)\bar{u} \cdot \bar{r}} \int_{4\pi} d\bar{\Omega} \frac{\hat{\Omega}^i}{1 + \nu'(s)\bar{u} \cdot \bar{\Omega}} .
\] (37)

where \(\hat{\Omega}^i\) is the projection of the vector \(\hat{\Omega}\) along the \(i\)th direction. We skip the geometrical evolution of the integral over \(\hat{\Omega}\) since it is developed widely in Pomraning (1973). After carrying out the integration over \(\hat{\Omega}\), we get the \(i\)th component of \(F\) as a function of \(E\):

\[
F'^{(i)}(s, \bar{r}) = \frac{c(c'(s) - 1)}{\nu'(s)} \int d\bar{u} f(\bar{u}) e^{\sigma'(s)\nu'(s)\bar{u} \cdot \bar{r}} \bar{u}^i
\]

\[
= \frac{c(c'(s) - 1)}{\sigma'(s)\nu'^2(s)} \frac{\partial}{\partial x_i} \int d\bar{u} f(\bar{u}) e^{\sigma'(s)\nu'(s)\bar{u} \cdot \bar{r}} .
\] (38)
Equation (38) is the $i$th component of the vector relationship of a modified Fick's law of this kind:

$$\vec{F}'(s, \vec{r}) = -cD'(c'(s), \sigma'(s)) \cdot \vec{\nabla}E'(s, \vec{r}), \quad (39)$$

with a modified $s$-dependent diffusion coefficient:

$$D'(c'(s), \sigma'(s)) \equiv \frac{1 - c'(s)}{\sigma'_0(s) \sigma'(s)}. \quad (40)$$

Substituting the modified cross-section and albedo into Equation (40) yields this diffusion coefficient:

$$D'(c'(s)) = c \frac{s}{(c\sigma + s)^2 \sigma'_0(s)} \equiv \frac{cD_0(c'(s))}{c\sigma + s}. \quad (41)$$

Now we compare the resulting modified $s$-dependent diffusion coefficient in Equation (41) with Equation (22) and solve for $A$ and $B$, by using the tabulated (or approximated) value of $\sigma'_0(s)$ or $D_0(c'(s))$ (Winslow, 1968; Heizler, 2010; Case et al., 1953). Since this value is involved, we expand the modified diffusion coefficient in a Taylor series:

$$\frac{c}{D'(c'(s))} = \frac{c\sigma + s}{D_0(c'(s))} \approx Bc\sigma + As + \mathcal{O}(s^2) + \cdots \quad (42)$$

recalling that $s \to 0$ corresponds to $t \to \infty$ according to the final-value theorem. That means an asymptotic treatment for the time, in addition to the derivation on space.

Using the numerical evaluation of $\sigma'_0(s)$ in Case et al. (1953) we plot in Figure 1 $A(\bar{\omega}_{\text{eff}})$ and $B(\bar{\omega}_{\text{eff}})$ as a function of the ratio between the material energy density and the radiation energy density, $\bar{\omega}_{\text{eff}}$. We see that $B(\bar{\omega}_{\text{eff}})$ is equal to the inverse of the asymptotic diffusion coefficient (Heizler, 2010; Case and Zweifel, 1967; Case et al., 1953): At the thin limit, $\bar{\omega}_{\text{eff}} \to 0$ ($U_m \to 0$), $B \to 1$, and in the diffusion (optically thick) limit $\bar{\omega}_{\text{eff}} \to 1$ ($E \to U_m$), $B \to 3$, as in the classic $P_1$ and diffusion approximations. Moreover, in $\bar{\omega}_{\text{eff}} \to 0$ the asymptotic $P_1$ approximation reproduced the $P_{1/3}$ approximation, $A \to 1$. As $\bar{\omega}_{\text{eff}}$ increases, the value of $A$ decreases so in the diffusion limit $\bar{\omega}_{\text{eff}} \to 1$, $A \approx 0.6$. Thus, the asymptotic $P_1$ approximation yields some physical base to the $P_{1/3}$ approximation; exact physical base for $\bar{\omega}_{\text{eff}} = 0$ and partial with $\bar{\omega}_{\text{eff}}$ increases. Anyway, it is surly much better than setting $A = 3$, the classic $P_1$ choice. We note that the value of $A$ can be evaluated for the case of $E < U_m$ ($\bar{\omega}_{\text{eff}} > 1$), so for $\bar{\omega}_{\text{eff}} \to \infty$, $A \approx 0.48$. The connection between the value of $A$ and the particle velocity will be discussed in Section IV.

The derivation so far is a full non-LTE derivation. However we need to evaluate $\bar{\omega}_{\text{eff}}$ before we actually solve the RTE (and the material equation)
for yielding $E$ and $U_m$. The practical way to use this asymptotic approximation in non-LTE codes is to evaluate $\tilde{\omega}_{\text{eff}}$ using $E$ and $U_m$ for each numerical cell from the previous time-step, like in multigroup calculations in neutronics or radiative diffusion. Because of that, the expansion of the asymptotic $P_1$ approximation to a multigroup (with several energy groups) case (Winslow, 1968) or to (linearly anisotropic) scattering problems (Winslow, 1968; Olson et al., 2000) is straightforward. This approximation does not suffer from the gradient-depending problems in a curvilinear meshes, in contrast to Flux-Limiters or Variable Eddington-Factor approximations (or any other gradient-dependent approximation).

A. Specific Case: LTE

For a full analytical derivation, we consider the LTE case (all the photons that are absorbed in the material are emitted immediately according to blackbody radiation). In this case, we assume that the material is in LTE with the radiation field ($E = U_m$). This is a quite accurate assumption for the asymptotic regime, checking for example the Su-Olson benchmark (Olson et al., 2000, Su and Olson 1996, 1997, 1999) (see also in Figure 4). We can clearly see that in long times, the radiation and the material energy curves are very close. Even in intermediate times they are quite the same and only in short times (near the source) is there a large difference between them.

Figure 1: The constants $A(\tilde{\omega}_{\text{eff}})$ and $B(\tilde{\omega}_{\text{eff}})$ for a general non-LTE problem as a function of the effective albedo (the ratio between the material energy density and the radiation energy density), $\tilde{\omega}_{\text{eff}}$, defined by Equation (24).
With this assumption, $\tilde{\omega}_{\text{eff}} = 1$ and the effective albedo is equal to:

$$c'(s) = \frac{\sigma}{\sigma'(s)} = \frac{1}{1 + \frac{s}{c\sigma}}.$$  \hfill (43)

Using the approximate expression for $D_0(c'(s))$ (Case et al., 1953):

$$D_0(c'(s) \approx 1) \approx \frac{1}{3} \left( 1 + \frac{4}{5}(1 - c'(s)) + \cdots \right),$$  \hfill (44)

substituting Equation (43) and using Equation (42), yields $B = 3$ and $A = 3/5$, exactly.

We note that for simplicity we can use the value of $A = 3/5$ and $B = 3$ in Equations (4) and (19) and (10). We call it a semi-LTE treatment since from one hand we still use different equations for the radiation energy and the material energy balance (and $E$ may differ from $U_m$), but the derivation of the coefficients $A$ and $B$ assumed LTE. This semi-LTE treatment can also be called, in the $P_{1/3}$ approximation notation, the $P_{1/5}$ approximation (the ratio of the two constants is $A/B = 1/5$). Another confirmation comes from Zimmerman’s note, concerning evaluating the Levermore-Pomraning Flux-Limiter diffusion coefficient (Olson et al., 2000), that for practical use, assuming $\tilde{\omega}_{\text{eff}} = 1$ (LTE case) yields a better solution than using the true $\tilde{\omega}_{\text{eff}}$ (again, just for the evaluation of the diffusion coefficient, not for setting $E = U_m$). This also gives some justification to use a semi-LTE treatment in our approximation, as defined.

B. Morel’s (Larsen’s) Asymptotics Accuracy Check

The new approximation is called the asymptotic $P_1$ (or asymptotic Telegrapher’s equation) approximation, since its derivation was based on assuming asymptotic solution on both space and time. One has to check if the asymptotic $P_1$ approximation satisfies the asymptotics accuracy check (at least) to the order $\mathcal{O}(1)$ in the diffusion limit. Since the asymptotic analysis of the exact RTE and the $P_{1/3}$ approximation is derived widely in Morel (2000), we briefly present the changes concerning the asymptotic $P_1$ approximation.

The first asymptotic $P_1$ equation is the conservation law (Equation (4)) and it is identical to the first equation of the $P_1$ and the $P_{1/3}$ approximations. Thus, it satisfies the diffusion limit to $\mathcal{O}(1)$. Converting $\partial/\partial t$ to $\epsilon^2 \partial/\partial t$ and $\nabla$ to $\epsilon \nabla$, expanding $E$ and $\tilde{F}$ in $\epsilon$ (see (Morel, 2000)), the second asymptotic $P_1$ equation (Equation (19)) yields:

$$\frac{A\epsilon^2}{c} \frac{\partial \tilde{F}}{\partial t} + \epsilon \epsilon \nabla E + \sigma B \tilde{F} = 0,$$  \hfill (45)

Since we know that both $P_1$ approximation ($A = 3$) and the $P_{1/3}$ approximation ($A = 1$) satisfy the diffusion limit to the $\mathcal{O}(1)$, (for the $P_1$ approximation, see, for example Shin et al., 1995), it is reasonable that any general $A(\tilde{\omega}_{\text{eff}})$ will also
satisfy the diffusion limit to the $O(1)$. Let’s check: The $O(0)$ equation arising from Equation (45) is:

$$\vec{F}^{(0)} = 0. \tag{46}$$

The $O(1)$ equation arising from Equation (45) is:

$$c\vec{E}^{(0)} + \sigma B \vec{F}^{(1)} = 0. \tag{47}$$

The $O(2)$ equation arising from Equation (45) is:

$$\frac{A}{c} \frac{\partial F^{(0)}}{\partial t} + c\vec{E}^{(1)} + \sigma B \vec{F}^{(2)} = 0. \tag{48}$$

Taking Equation (46) into account, Equation (48) becomes:

$$c\vec{E}^{(1)} + \sigma B \vec{F}^{(2)} = 0 \tag{49}$$

Equations (46), (47), and (49) agree with Equations (13), (18), and (21) in Morel (2000), taking $B = 3$ (which corresponds to $\omega_{eff} = 1$, $E = U_m$). Thus, the asymptotic $P_1$ approximation is correct through $O(1)$ in the equilibrium diffusion limit (when indeed the radiation energy is in equilibrium with the material energy) with any general $A(\omega_{eff})$, as expected (since the derivative with respect to time is proportional to $\epsilon^2$).

### IV. GREEN’S FUNCTION FOR THE ASYMPTOTIC $P_1$ EQUATIONS

In the previous sections we introduced the equations of the asymptotic $P_1$ equations for radiative transfer from an asymptotic derivation of the grey (mono-energetic), time-dependent Boltzmann equation. In this section we find analytically the Green function for the asymptotic $P_1$ equations for the LTE case, one-dimensional (and thus, setting $\partial/\partial z$ for $\vec{\nabla}$ when $z$ is the spatial coordinate) slab geometry, which is helpful for a general source (in specifically, a LTE-adjustment for the Su-Olson benchmark). The derivation starts for the semi-LTE case (assuming constant, time and space-independent $A$ and $B$) and we mark explicitly when a full LTE assumption is made (setting $E = U_m$).

First, we follow the McClarren et al. (2008) notation of dimensionless space and time variables:

$$x = \sigma z, \quad \tau = \epsilon c \sigma t, \tag{50}$$

and defining this dimensionless energy density, radiation flux and material energy density and radiation source:

$$\varepsilon \equiv \frac{E}{aT_r^4}, \quad U \equiv \frac{U_m}{aT_r^4} = \frac{T_m^4}{T_r^4}, \quad \vec{F} \equiv \frac{F}{aT_r^4}, \quad Q \equiv \frac{S}{\sigma aT_r^4}. \tag{51}$$
$T_r$ is a constant that is used to define dimensionless quantities (not to confuse with the radiation temperature $T_R$). With the definitions of Equations (50) and (51), Equations (4), (19), and (10) can be written in the one-dimensional, slab geometry case as:

\( \varepsilon \frac{\partial \mathcal{E}}{\partial \tau} + \frac{1}{c} \frac{\partial \mathcal{F}}{\partial x} = (U - \mathcal{E}) + Q \) (52a)

\( \varepsilon A \frac{\partial \mathcal{F}}{\partial \tau} + c \frac{\partial \mathcal{E}}{\partial x} + B \mathcal{F} = 0 \) (52b)

\( \frac{\partial U}{\partial \tau} = (\mathcal{E} - U) \) (52c)

Applying the Laplace transform on the time to Equations (52) using \( Q = \delta(x)\delta(t) \) yields:

\( \varepsilon \mathcal{S} \hat{\mathcal{E}} + \frac{1}{c} \frac{\partial \hat{\mathcal{F}}}{\partial x} = (\hat{U} - \hat{\mathcal{E}}) + \delta(x) \) (53a)

\( \varepsilon A \mathcal{S} \hat{\mathcal{F}} + c \frac{\partial \hat{\mathcal{E}}}{\partial x} + B \hat{\mathcal{F}} = 0 \) (53b)

\( \mathcal{S} \hat{U} = (\hat{\mathcal{E}} - \hat{U}) \) (53c)

Substituting Equation (53c) in Equation (53a) yields:

\( \varepsilon \mathcal{S} \hat{\mathcal{E}} + \frac{1}{c} \frac{\partial \hat{\mathcal{F}}}{\partial x} = \hat{\mathcal{E}} \left( \frac{1}{1+s} - 1 \right) + \delta(x) \) (54)

Equations (54) and (53b) are two closed equations for \( \hat{\mathcal{E}} \) and \( \hat{\mathcal{F}} \). Applying the Fourier transform on the space to Equations (54) and (53b) yields:

\( \varepsilon \mathcal{S} \mathcal{F} + \frac{ik}{c} \hat{\mathcal{F}} = \mathcal{S} \left( \frac{1}{1+s} - 1 \right) + 1 \) (55a)

\( \varepsilon A \mathcal{S} \hat{\mathcal{F}} + ikc \hat{\mathcal{E}} + B \hat{\mathcal{F}} = 0 \) (55b)

Substituting Equation (55b) in Equation (55a) and solving for \( \mathcal{S}(k, s) \) yields:

\( \mathcal{S}(k, s) = \frac{\mathcal{S} + B}{k^2 + (\varepsilon s^2 + \varepsilon s + s)(\varepsilon A s + B)/(1+s)} \) (56)

Equation (56) can be derived alternatively for the dimensionless equation of the asymptotic Telegrapher's equation (Equation (20)), which yields the same expression for \( \mathcal{S}(k, s) \).

Applying the inverse-Fourier transform to Equation (56) yields the \( s \)-dependent energy density:

\( \mathcal{S}(x, s) = \frac{\sqrt{(1+s)(\varepsilon A s + B)}}{2\sqrt{\varepsilon s^2 + \varepsilon s + s}} e^{-\sqrt{\frac{(\varepsilon s^2 + \varepsilon s + s)(\varepsilon A s + B)}{1+s}} |x|} \) (57)
So far the derivation was exact (again, assuming constant $A$ and $B$) and Equation (57) is a direct generalization of McClarren et al. (2008) ($A = B = 3$) with two general constants, $A$ and $B$ (however, $s$-dependent). To our best knowledge, the inverse-Laplace transform cannot be applied over Equation (57) analytically (using two general constants, $A$ and $B$), as was done in the classic $P_1$ case (McClarren et al., 2008). To find the energy density $\mathcal{E}(x, t)$ one can solve the inverse-Laplace transform numerically (for example, see Abate and Valko, 2004). Another way, that is taken in this study is to make some additional approximations to Equation (57) to get an approximate analytic solution. First, we set $\varepsilon = 1$, which means that the material heat capacity equals the radiation heat capacity (see Equation (18)). This assumption appears also in McClarren et al. (2008). Second, since our derivation is asymptotic, both in space and time, we evaluate the $s$-dependent expressions in Equation (57) for small $s$ (according to final-value theorem). Therefore, the derivation from now on is not a direct generalization of McClarren et al. (2008). Under these assumptions Equation (57) reduces to:

$$
\mathcal{E}(x, s) \approx \frac{As + B}{2\sqrt{2s(As + B)}} e^{-\sqrt{2s(As + B)|x|}}.
$$

As a matter of fact, Equation (58) is a full-LTE derivation of the asymptotic $P_1$ equations, when the material and the radiation can be expressed in “one-temperature” notation (as for the diffusion approximation in Section II). Adding Equation (19) with Equation (10), assuming the LTE assumption $E \approx U_m$, and applying the Laplace transform (to the dimensionless equation) yields:

$$
(1 + \varepsilon)s\mathcal{E} + \frac{1}{c} \frac{\partial \mathcal{E}}{\partial x} = \delta(x).
$$

Applying the Fourier transform to Equation (59) and solving for $\mathcal{E}(k, s)$ yields

$$
\mathcal{E}(k, s) = \frac{\varepsilon As + B}{k^2 + (1 + \varepsilon)s(\varepsilon As + B)}.
$$

Applying the inverse-Fourier transform to Equation (60) yields the LTE notation for $\mathcal{E}(x, s)$:

$$
\mathcal{E}(x, s) = \frac{\varepsilon As + B}{2\sqrt{(1 + \varepsilon)s(\varepsilon As + B)}} e^{-\sqrt{(1 + \varepsilon)s(\varepsilon As + B)|x|}},
$$

which is identical to Equation (58), assuming $\varepsilon = 1$. Thus, we denote the analytic solution for the asymptotic $P_1$ approximation presented here by the LTE asymptotic $P_1$ solution. Choosing $A = B = 3$ is in the LTE $P_1$ solution (for $\varepsilon = 1$, and is different from the non-LTE $P_1$ solution found by McClarren et al., 2008).
Applying the inverse-Laplace transform to Equation (58) (or Equation (61) with $\varepsilon = 1$) is available analytically (with a simple shift in space of the source to $x_0$, i.e. $\delta(x) \rightarrow \delta(x - x_0)$) (Roberts and Kaufman, 1966):

$$G(x, x_0, \tau) = \frac{1}{2\sqrt{2A}} e^{-\frac{B}{2\tau}} \left[ A\delta(\tau - \sqrt{2A}|x - x_0|) + B \right] \cdot I_0 \left( \frac{B}{2A} \sqrt{\tau^2 - 2A(x - x_0)^2} \right)$$

where $I_n(x')$ is the modified-Bessel function of first kind of order $n$ and $H(x, \tau)$ is the Heaviside-step function. Expressing explicitly the derivatives in Equation (62) yields:

$$G(x, x_0, \tau) = \frac{1}{2\sqrt{2A}} e^{-\frac{B}{2\tau}} \left[ A\delta(\tau - \sqrt{2A}|x - x_0|) + B \right] \cdot I_0 \left( \frac{B}{2A} \sqrt{\tau^2 - 2A(x - x_0)^2} \right)$$

Equation (63) is the LTE asymptotic $P_1$ solution for $A = 3/5$ and $B = 3$, and the LTE $P_1$ solution for $A = 3$ and $B = 3$. This LTE solution is more similar to the Green function of the neutron transport $P_1$ solution (Heizler, 2010; Krainer and Pucker, 1969; Ganapol and Grossman, 1973; Ganapol, 1980) than the non-LTE classic $P_1$ solution, given in McClarren et al. (2008):

$$G_{\text{grey}, P_1}(x, x_0, \tau) = \frac{3}{2} e^{-\tau} \left[ \frac{\tau I_1(\sqrt{\tau^2 - 3(x - x_0)^2})}{\sqrt{\tau^2 - 3(x - x_0)^2}} \cdot H(\tau - \sqrt{3}|x - x_0|) \right]$$

In addition, we introduce the LTE diffusion solution (Equation (16)) for the Green function, resulting from a similar derivation, using integral transforms:

$$G_D(x, x_0, \tau) = \frac{1}{2\sqrt{2\pi\tau}} e^{-\frac{B}{2\pi\tau}(x - x_0)^2}. \quad (65)$$

We can immediately see that in contrast to the infinite particle velocity of the diffusion approximation (Equation (65)) and the wrong finite velocity of the non-LTE classic $P_1$ approximation, $c/\sqrt{3} \approx 0.58c$, the LTE asymptotic $P_1$ approximation yields almost the correct particle velocity, $\sqrt{5/6}c \approx 0.91c$. 

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The LTE classic $P_1$ approximation prediction is in a poor agreement to the particle velocity, $c/\sqrt{6} \approx 0.41c$. In Figure 2 we can see the Green functions for the different approximations (without the delta-component, which decays in any case faster than other components (McClarren et al. 2008)).

At small $\tau$ the difference between the approximations is quite large, while the diffusion approximation and the asymptotic $P_1$ approximation are close,
Asymptotic Telegrapher’s Equation (P₁) Approximation

V. ASYMPTOTIC P₁ LTE SOLUTION TO THE SU-OLSON BENCHMARK PROBLEM

Using the Green function that was found in Section IV, we can find the LTE solution of the asymptotic P₁ approximation for the Su-Olson benchmark (Olson et al., 2000; Su and Olson, 1996, 1997, 1999). The source in this benchmark is a constant source in the range of \(|x| \leq 0.5\), operating in the time-interval \(0 \leq \tau \leq 10\).

The energy density in that case using Equation (63) is for \(\tau \leq 10\):

\[
\varnothing(x, \tau) = \frac{1}{2\sqrt{2A}} \int_{-0.5}^{0.5} dy \left[ Ae^{-\frac{B}{2\sqrt{2A}}|x-y|} H(\tau - \sqrt{2A}|x-y|) \right. \\
+ \left. \frac{B}{2} \int_{0}^{\tau} d\tau' e^{-\frac{B}{2\sqrt{2A}}\tau'} H(\tau' - \sqrt{2A}|x-y|) \right] I_0 \left( \frac{B}{2\sqrt{2A}} \sqrt{\tau^2 - 2A(x-y)^2} \right)
\]

and for \(\tau > 10\):

\[
\varnothing(x, \tau) = \frac{1}{2\sqrt{2A}} \int_{-0.5}^{0.5} dy \left[ Ae^{-\frac{B}{2\sqrt{2A}}|x-y|} H(\tau - \sqrt{2A}|x-y|) \right. \\
+ \left. \frac{B}{2} \int_{10}^{\tau} d\tau' e^{-\frac{B}{2\sqrt{2A}}\tau'} H(\tau' - \sqrt{2A}|x-y|) \right] I_0 \left( \frac{B}{2\sqrt{2A}} \sqrt{\tau^2 - 2A(x-y)^2} \right)
\]

Equations (66) and (67) are similar to the solutions in McClarren et al. (2008), with the appropriate Green function (assuming LTE). The radiation flux \(F\) and the material energy \(U\) can be developed easily from the energy density \(\varnothing\), using Equations (52).

The integrals in Equations (66) and (67) can be solved numerically. The solution of the energy density for non-LTE transport (from Su and Olson, 1997), the LTE diffusion (using Equation (65) with \(B = 3\)), the LTE P₁ approximation (Equation (66) with \(A = B = 3\)), and the asymptotic P₁ approximation (Equation (66) with \(A = 3/5\) and \(B = 3\)), are shown in Figure 3.

except the wave front zone. At larger \(\tau\), all the approximations are closer and differ only in the wave front.
Figure 3: The radiation energy in the asymptotic \( P_1 \) approximation (Equation (66) with \( A = 3/5 \) and \( B = 3 \)), along with the non-LTE transport (from Su and Olson, 1997), the LTE diffusion (using Equation (65) with \( B = 3 \)) and the classic LTE \( P_1 \) approximation (Equation (66) with \( A = B = 3 \)): (a) Linear scale and (b) logarithmic scale.

We can see that near the source regime (with \( \tau > 1 \)), all the approximations yield similar accuracy. Far from the source, for intermediate times (\( \tau \approx 1 \)), near the wave-front area (see especially Figure 3(b)), the asymptotic \( P_1 \) approximation yields the best LTE approximation to the exact transport solution. The diffusion approximation yields an infinite particle velocity while the classic \( P_1 \) approximation yields the wrong finite velocity \( (c/\sqrt{6} \approx 0.41c) \). The asymptotic \( P_1 \) approximation yields almost the correct particle-velocity \( (\sqrt{5}/6c \approx 0.91c) \), thus it yields the best approximate solution near the wave front. For long times (\( \tau > 3 \)), the LTE diffusion approximation and the asymptotic \( P_1 \) approximation yields similar results. All the LTE approximations yield poor accuracy with the transport solution inside the source, especially in early times, where (and also
when) there is still a large difference between the material and the radiation energies (see Figure 4). The fact that the LTE $P_1$ approximation yields worse results than the diffusion approximation is not surprising, due to the well-known fact that the $P_1$ approximation is less accurate than the time-dependent diffusion equation, despite its hyperbolic nature; it over estimates the flux-limiting (for example, see Bengston, 1958).

Moreover, the asymptotic $P_1$ approximation yields a good approximation also in comparison to non-LTE approximations. In Figure 4 we can see the material and radiation energies of the transport solution (from Su and Olson, 1997), along with the non-LTE $P_1$ approximation (Equation (64), taken from McClarren et al., 2008), and the LTE asymptotic $P_1$ approximation.
Figure 5: The radiation energy in the LTE asymptotic $P_1$ approximation along with the radiation energy of the transport solution, the $P_{1/3}$ approximation (both taken from Su and Olson, 1997), and the $P_1$ approximation (Equation (64), taken from McClarren et al., 2008), in a logarithmic scale.

From Figure 4 we can see that except the source area itself, where LTE assumption breaks down, the LTE asymptotic $P_1$ approximation yields a better approximation than the non-LTE $P_1$ approximation. In the wave front area the situation is clear (see Figure 4(b)) since the LTE asymptotic $P_1$ yields a better particle velocity. But although the asymptotic $P_1$ approximation was derived in the asymptotic regime (i.e. $t \to \infty$ and far from sources), the results for the energy density yield better approximation also for $|x| > 0.5$ and for intermediate times (i.e. $\tau > 1$) far from the wave front, than the non-LTE $P_1$ approximation.

A slightly fortuitous result is that the LTE asymptotic $P_1$ approximation yields a solution better than the non-LTE $P_{1/3}$ approximation (especially for intermediate times, $\tau = 1, \tau = 3.16$), again, except in the source area itself ($|x| \leq 0.5$). The (numeric) solutions of the non-LTE $P_{1/3}$ approximation (from Olson et al., 2000) are shown in Figure 5, along with the transport solution (from Su and Olson, 1997), the non-LTE $P_1$ solution (Equation (64)), taken from McClarren et al., 2008) and the LTE asymptotic $P_1$ solution. We can see in Figure 5 that except the wave front itself (the $P_{1/3}$ approximation yields an exact particle velocity) when the energy density is completely negligible, the LTE asymptotic $P_1$ approximation (i.e., a $P_{1/5}$ approximation) clearly yields a better approximation than the $P_{1/3}$ approximation (again, outside the source regime). For long times, all the approximations are similar.

VI. DISCUSSION

The asymptotic $P_1$ approximation (or the asymptotic Telegrapher’s equation approximation) is derived for the case of radiative transfer from an asymptotic
derivation of the exact multidimensional general geometry, time-dependent mono-energetic Boltzmann equation for radiative transfer, both on space and time, in a similar way that it is developed for neutronics (Heizler, 2010). The resulting approximation yields the exact asymptotic eigenvalue of the steady-state time-independent Boltzmann equation, and the (almost) correct particle velocity, which governs the wave front, and thus, gives some physical base to the $P_{1/3}$ approximation, exact for $\tilde{\omega}_{\text{eff}} = 0$ and partial with $\tilde{\omega}_{\text{eff}}$ increases.

An analytic Green function for the asymptotic $P_1$ approximation is calculated in the same techniques used in a previous study (McClarren et al., 2008), for the case of an equal material and radiation heat-capacities ($\varepsilon = 1$), and in the limit of local thermodynamic equilibrium (LTE). For a more general case, we can use a numeric inverse Laplace transform or direct numerical simulations for solving $P_1$ equations.

We test the asymptotic $P_1$ approximation by the Su-Olson benchmark (Olson et al., 2000; Su and Olson, 1996, 1997, 1999), using the given Green function. We find that the LTE solution of the asymptotic $P_1$ solution is the best LTE solution of the offered approximation, especially in modeling the wave front, yielding the almost correct particle velocity. The LTE solution of the asymptotic $P_1$ solution was found to be more accurate even from non-LTE approximations, including the classic $P_1$ approximation and the $P_{1/3}$ approximation (at least for intermediate times), in the out-of-source regime. Even with this simple LTE approach, we reproduce the full transport solution with a good agreement, including the wave front. Yet, general conclusions might best be saved for more detailed comparisons, especially testing the non-LTE asymptotic $P_1$ approximation.

In the future work we plan to develop a numerical simulation for solving full non-LTE solution using the asymptotic $P_1$ approximation. With such simulations we will test our approximation in complex problems including space-dependent opacity and optically thin media.

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REFERENCES


Asymptotic Telegrapher’s Equation ($P_1$) Approximation


